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Rev. T. E. R. PHILLIPS, PRESIDENT, in the Chair.

The election by the Council of the following Fellows and Associates was duly confirmed :—

FELLOWS.

James Currie, M.A., Martnaham, Cardonald, Glasgow, S.W. 2 ;
Wilfred Denne, Dingle Cottage, Sheepcote Lane, Watford, Herts ;
Charles W. Elmer, 15 Park Row Building, New York City, U.S.A. ; and
Harold Henry Thorne, The University of Sydney, Sydney, N.S.W. ; and 68 Lambton Road, London, S.W. 20.

ASSOCIATES.

Dr. Edwin Hubble, Mount Wilson Observatory, California, U.S.A. ;
General Gustave Ferrié, Directeur des Services de la Télégraphie Militaire, Boulevard Latour, Maubourg, Paris ; and
Captain Leon Herrero, San Fernando Observatory, Spain.

Sixty-seven presents were announced as having been received since the last meeting, including, amongst others :—

Abbott (Mrs. J. R.) : Framed portrait of Copernicus.
Bacon (Gertrude) : Memories of Land and Sky.
Jeans (J. H.) : Astronomy and Cosmogony.
Schlesinger (Frank) : General Catalogue of Stellar Parallaxes.

Professor W. H. Wright delivered the George Darwin Lecture, taking as his subject " Photography of the Planets in Light of Different Wave-Lengths."

A Remainder Formula and its Use in the Development of the Disturbing Function by Harmonic Analysis. By Ernest W. Brown, Sc.D., F.R.S.

1. The calculation of the coefficients in the expansion of a known oscillating function by numerical harmonic analysis has many advantages over methods which depend on expansion formulæ, but its lack of theoretical interest and the fact that its development depends on quite elementary mathematics have tended to prevent detailed descriptions from appearing in text-books or memoirs. Much depends on the mere mechanical arrangement of the work, and a study of this is not usually undertaken unless the number of such analyses to be made is large. When the arrangement is in good order, several of the usual sources of error are avoided, the results themselves constitute tests of accuracy, and the work is easily adapted to simple routine computations.

Double harmonic analysis, *i.e.* analysis in which the angles are of the form $i\theta \pm i'\theta'$ (i, i' integers) has been generally avoided,* and yet it possesses the advantage that the work can be reduced to a number of similar single analyses. When these are carried out in parallel columns, the time necessary to perform eight or ten of them is but little more than twice that required for one.

2. The main object of this paper is the reduction of the disturbing function to a form which permits the expansion to be reduced to one double harmonic analysis. In order to show the nature of the problem and the method of solution, let us consider the function

$$(1 - 2\alpha \cos \psi + \alpha^2)^{-\frac{1}{2}},$$

where α is a numerical constant which in most cases lies between ·4 and ·7. Well-known formulæ are available for its expansion into a Fourier series. But if we wish to find the coefficients by harmonic analysis we are faced by a practical difficulty: we need the earlier harmonics only, but we need them to five or more places of decimals, and to get them to this degree of accuracy some fifteen or more special values of ψ must be used. If, however, a sufficiently brief formula for the remainder after, say, nine terms could be found we could analyse

$$(1 - 2\alpha \cos \psi + \alpha^2)^{-\frac{1}{2}} - \text{“ rem.”}$$

with only nine special values of θ and thus obtain the required coefficients accurately.

Actually, $\cos \psi$ is of the form

$$\cos \psi = \mathrm{P} \cos (\theta - \theta') + \mathrm{Q} \sin (\theta - \theta'),$$

* Leverrier (*Par. Obs. Mém.*, vol. i., Add. i.) carried out an analysis of this kind for the mutual perturbations of Jupiter and Saturn. He used 32 divisions of the mean anomaly of Jupiter, and 16 of that of Saturn, the work being done with the complete disturbing function. Even so, the numerical results have not the accuracy needed for the long-period terms.

and α, P, Q are themselves periodic functions of θ, θ' such that a multiple i of θ in an angle is always accompanied by a factor e^i in the coefficient, e being a small parameter of the order ·1 ; and similarly for multiples of θ'. The series due to the expansions of α, P, Q converge with sufficient rapidity in general for the efficient use of harmonic analysis, and hence attention is mainly to be directed towards multiples of $\theta - \theta'$ or ψ.

3. *Summary.*—The first part contains a formula for the remainder after a given number of terms of the function $(1 - 2\alpha \cos \psi + \alpha^2)^{-s}$ expanded into a cosine Fourier series with argument ψ. It is next shown how the disturbing forces are obtained. The calculation of the special values of the functions needed is indicated. Finally, an outline of the method of performing the double harmonic analysis is sketched with a consideration of the errors of the coefficients when sixteen special values are used for each argument. Details are mentioned when they are of importance for the abbreviation of the computations.

4. *The Coefficients expressed as Definite Integrals.*—The required formulæ are obtained briefly through the use of the definite integral

$$\frac{1}{\pi}\int_0^{\pi} \frac{\sin^{2i+2j}\beta}{\cos^{2j}\beta} d\beta = \frac{(2i+2j-1)(2i+2j-3)\ \ldots\ (2j+1)}{(2i)\qquad (2i-2)\quad \ldots\quad 2}(-1)^j, \quad (4.1)$$

in which $i > 0$, j is a positive or negative integer or zero, and the singular point $\beta = \pi/2$ is excluded from the range of integration ; when $i = 0$, we put unity for the right-hand member.

If then

$$\mathrm{R} \equiv (1 - 2\alpha \cos \psi + \alpha^2)^{-s} \equiv \tfrac{1}{2}a_0 + \Sigma_1^{\infty} a_i \cos i\psi, \quad . \quad (4.2)$$

a well-known formula * for a_i, expanded in powers of α^2, shows that

$$a_i = (-1)^{s-\frac{1}{2}}\alpha^i \frac{2}{\pi}\int_0^{\pi} \frac{\sin^{2i+2s-1}\beta}{\cos^{2s-1}\beta} \frac{d\beta}{(1-\alpha^2\sin^2\beta)^s}, \quad . \quad (4.3)$$

This is a convenient extension of a formula which is known † for the case $s = \frac{1}{2}$.

Further, if we put $\alpha^2 = p/(p+1)$ under the integral sign and expand in powers of p, we obtain still another method of deducing the series used in my previous paper ‡ on this subject.

5. *A Formula for the Remainder after* i *Terms.*—Put

$$x = \text{exp. } \psi, \qquad 2\cos i\psi = x^i + x^{-i}, \quad . \quad . \quad (5.1)$$

so that R is the real part of $\frac{1}{2}a_0 + \Sigma_1^{\infty} a_i x^i$ and the remainder after i terms is the real part of $\Sigma_i^{\infty} a_i x^i$. This latter expression, after the

* See, *e.g.*, Tisserand, *Mécan. Cél.*, vol. i., p. 274.

† Tisserand, *loc. cit.*, p. 272.

‡ *Monthly Notices*, 1928 March. The following corrections may be recorded here : p. 460, line 7, the sign of the second term on the right is minus ; p. 462, line 7, *for* test *read* last ; §§ 6, 8, put a bar over the letter p to distinguish it from the same letter used elsewhere in a different sense.

substitution of (4.3) for a_i and the summation of a geometric series, becomes

$$2(-1)^{s-\frac{1}{2}}\alpha^i x^i \frac{1}{\pi}\int_0^\pi \frac{\sin^{2i+2s-1}\beta}{\cos^{2s-1}\beta}\cdot\frac{1}{1-\alpha x\sin^2\beta}\cdot\frac{d\beta}{(1-\alpha^2\sin^2\beta)^s}. \quad (5.2)$$

It is evident that we can also deduce a similar formula for the sum of any number of consecutive terms.

In order to obtain (5.2) in a form convenient for calculation, let us apply the bilinear transformation

$$q = \frac{x-\alpha}{1-\alpha x} \equiv \frac{(1+\alpha^2)\cos\psi - 2\alpha + \iota(1-\alpha^2)\sin\psi}{1-2\alpha\cos\psi+\alpha^2} \equiv \cos\phi + \iota\sin\phi, \quad (5.3)$$

this definition of ϕ being possible since $|q| = 1$. We deduce

$$x = \frac{q+\alpha}{1+\alpha q}, \qquad 1-\alpha x = \frac{1-\alpha^2}{1+\alpha q}. \quad (5.4)$$

If, further, we put

$$p = \frac{\alpha^2}{1-\alpha^2}, \qquad \alpha^2 = \frac{p}{1+p}, \quad (5.5)$$

a few simple operations will give (5.2) in the form

$$2\frac{\alpha^i x^i}{1-\alpha x}\frac{(-1)^{s-\frac{1}{2}}}{(1-\alpha^2)^s}\frac{1}{\pi}\int_0^\pi \frac{\sin^{2i+2s-1}\beta}{\cos^{2s-1}\beta}\cdot\frac{1}{1+p\cos^2\beta+\frac{pq}{\alpha}\cos^2\beta}\cdot\frac{d\beta}{(1+p\cos^2\beta)^s}. \quad (5.6)$$

6. This last expression can be expanded in various ways. If we put

$$\frac{s(s+1)\ldots(s+i-1)}{1.2\ldots i}\mathrm{P} \equiv (-1)^{s-\frac{1}{2}}\frac{1}{\pi}\int_0^\pi \frac{\sin^{2i+2s-1}\beta}{\cos^{2s-1}\beta}\frac{d\beta}{(1+p\cos^2\beta)^{s+1}},$$

so that, on expansion in powers of p, and with the use of (4.1), we obtain

$$\mathrm{P} = 1 - \frac{1^2-s^2}{i+1}\cdot\frac{p}{1} + \frac{(1^2-s^2)(2^2-s^2)}{(i+1)(i+2)}\cdot\frac{p^2}{1.2} - \ldots, \quad (6.1)$$

then (5.2) or (5.6) becomes

$$\frac{2\alpha^i x^i}{(1-\alpha x)(1-\alpha^2)^s}\frac{s(s+1)\ldots(s+i-1)}{1.2\ldots i}\left\{\mathrm{P} + \frac{pq}{\alpha}\cdot\frac{1}{s+1}\frac{\partial\mathrm{P}}{\partial p} + \left(\frac{pq}{\alpha}\right)^2\cdot\frac{1}{(s+1)(s+2)}\cdot\frac{\partial^2\mathrm{P}}{\partial p^2} + \ldots\right\}. \quad (6.2)$$

Finally, we need the real part of (6.2). Since (5.3) gives the real and imaginary parts of q, and since

$$\frac{x^i}{1-\alpha x} = \frac{\cos i\psi - \alpha\cos(i-1)\psi}{1-2\alpha\cos\psi+\alpha^2} + \iota\frac{\sin i\psi - \alpha\sin(i-1)\psi}{1-2\alpha\cos\psi+\alpha^2} \equiv \mathrm{C} + \iota\mathrm{S}, \quad (6.3)$$

the remainder after i *terms in the Fourier expansion of* R *is given by*

$$2 \cdot \frac{s(s+1) \ldots (s+i-1)}{1 \cdot 2 \ldots i} \cdot \frac{\alpha^i}{(1-\alpha^2)^s}(CQ_c - SQ_s), \quad . \quad (6.4)$$

where C, S *are defined in* (6.3), P *in* (6.1), *and* Q_c, Q_s *by the equations*

$$Q_c \equiv P + \frac{p}{\alpha} \cdot \frac{\cos\phi}{s+1} \frac{\partial P}{\partial p} + \frac{p^2}{\alpha^2} \frac{\cos 2\phi}{(s+1)(s+2)} \frac{\partial^2 P}{\partial p^2} + \ldots, \quad (6.5)$$

$$Q_s \equiv \frac{p}{\alpha} \cdot \frac{\sin\phi}{s+1} \frac{\partial P}{\partial p} + \frac{p^2}{\alpha^2} \frac{\sin 2\phi}{(s+1)(s+2)} \frac{\partial^2 P}{\partial p^2} + \ldots . \quad (6.6)$$

7. Since these forms are devised for calculation by harmonic analysis, they are used by inserting numbers for the various symbols. Their suitability depends mainly on the rapidity with which the series for P, Q_c, Q_s may converge. It appears that the value $i = 9$ is the best choice and the only values of s needed (§ 12) will be $\frac{1}{2}$, $\frac{3}{2}$. Most of the required values of p will lie between ·5, 1·0. Corresponding values of α, p, p/α are shown in the following table :—

α	·30	·40	·50	·60	·70	·75
p	·10	·19	·33	·56	·96	1·29
p/α	·33	·48	·67	·94	1·37	1·71

8. With these facts in view, the following series have been computed :—

Series for P with $i = 9$, $s = \frac{1}{2}$.

+1·000 000	+·965 359	+·935 378
− ·075 000p	−·064 159$(p-·5)$	−·056 13 $(p-1)$
+ ·012 784p^2	+·009 21 $(p-·5)^2$	+·007 02 $(p-1)^2$
− ·003 107p^3	−·001 80 $(p-·5)^3$	−·001 18 $(p-1)^3$
+ ·000 941p^4	+·000 50 $(p-·5)^4$	+·000 24 $(p-1)^4$
− ·000 333p^5	−·000 2 $(p-·5)^5$	−·000 07 $(p-1)^5$
+ ·000 13 p^6		

$$\frac{s(s+1) \ldots (s+8)}{1 \cdot 2 \ldots 9} = \frac{12155}{65536}.$$

Series for P with $i = 9$, $s = \frac{3}{2}$.

+1·000 000	+1·060 221	+1·116 546
+ ·125 000p	+ ·116 250$(p-·5)$	+ ·109 292$(p-1)$
− ·009 943p^2	− ·007 729$(p-·5)^2$	− ·006 278$(p-1)^2$
+ ·001 864p^3	+ ·001 169$(p-·5)^3$	+ ·000 80 $(p-1)^3$
− ·000 493p^4	− ·000 242$(p-·5)^4$	− ·000 14 $(p-1)^4$
+ ·000 160p^5	+ ·000 06 $(p-·5)^5$	+ ·000 03 $(p-1)^5$
− ·000 060p^6	− ·000 02 $(p-·5)^6$	

$$\frac{s(s+1) \ldots (s+8)}{1 \cdot 2 \ldots 9} = \frac{230945}{131072}.$$

The calculation of these series has been carried out with proper precautions to secure their numerical accuracy. The terms in the series in powers of p were computed as far as p^{16} and by induction so as to include several more powers, in order to secure the necessary accuracy for the series in powers of $p - 1$. As soon as P, $\partial P/\partial p_0$ have been obtained for $p = p_0$, the remaining derivatives needed to obtain the expansions in powers of $p - p_0$ are obtainable from recurrence formulæ deduced from the properties of the hypergeometric series of which P is a particular case. The earlier derivatives were also obtained from the series in powers of p. Comparisons of the values deduced from the three expansions for particular values of p were also made.

From these series it is easy to compute series in powers of $p - p_0$ where $p_0 = \cdot 3, \cdot 4, \ldots 1 \cdot 2$, and with such series available it will be rare that more than two terms would be needed in the calculation of P for any given value of a. The numerical results are sufficient for the calculation of the disturbing function to eight or nine significant figures : in most cases six will be sufficient.

9. The series for Q_c, Q_s, like all these series, converge rapidly at first, but the convergence is less rapid when higher accuracy is needed. The calculations for any particular case will be much lessened if, with the tables for P, we include the values of $\partial P/\partial p$ at intervals of $0 \cdot 1$ for p and form a double entry table for

$$\frac{100 \cos 2\phi}{(s+1)(s+2)} \frac{\partial^2 P}{\partial p^2} + \frac{100 \cos 3\phi}{(s+1)(s+2)(s+3)} \cdot \frac{p}{a} \frac{\partial^3 P}{\partial p^3} + \ldots,$$

and for the similar series in which $\sin j\phi$ replaces $\cos j\phi$. In these tables, three places of decimals will suffice. To avoid the use of second differences, they should be tabulated for intervals of $0 \cdot 1$ for p and 1° for ϕ. Some five to ten thousand entries would be required ; included in the estimation is the printing of the first differences or variations for both p, ϕ. The tables would be computed for $i = 9$ with $s = \frac{1}{2}, \frac{3}{2}$.

Theoretically, the series for Q_c, Q_s are absolutely convergent only when $p/a < 1$. But, since the functions which they represent have no singular points as long as p/a is finite, the series can be used as they stand for all values of p/a likely to be required. Even if it were not so, it would be possible to replace them by series in powers of $p/a - p_0/a_0$, where $p_0/a_0 < 1$, which would fulfil all needs. It is a curious fact that the larger a_0 is in these transformed series, the more rapid is the convergence.

10. The idea which has prompted the developments in this and in my previous paper (*loc. cit.*, *ante*) has been the replacement of a slowly converging series by the product of a finite function and a rapidly converging series. The latter can then either be used directly without extensive computations ; or tables in which only first differences are needed can be formed, such tables being quite limited in extent. Several modern developments are available for use. Computing machines and the simplification of routine calculations which can be performed by those having little more than a school education have made the formation of tables mainly a question of cost, while the typewriter and the repro-

ductions of typewritten matter by photographic methods which are as legible as printed matter have greatly diminished the labour of proof-reading and the cost of printing.

On the theoretical side, the methods of this paper enable one to estimate with but little calculation the order of magnitude of any coefficient, because it is at once evident from the numerical developments that, particularly for large values of i, the magnitude depends mainly on the finite function and is but little affected by the series. A rough approximation can in fact be obtained by putting $P = Q_c = 1$, $Q_s = 0$ when i is not too small.

11. *Calculation of the Disturbing Forces.*—Let r, r' be the radii vectores of the two planets, ψ the angle between r, r', and Δ their distance apart. Then

$$\Delta^2 = r^2 + r'^2 - 2rr' \cos\psi \equiv r'^2(1 + \rho^2 - 2\rho\cos\psi) \equiv r'^2\Delta_0^2, \quad (11.1)$$

where $\rho \equiv r/r'$ with * $r < r'$, accented letters referring to the disturbing planet. For the disturbed planet, let v be the true longitude reckoned in the usual manner, f the true anomaly, l the mean anomaly, $2a$ the major axis, e the eccentricity, ϖ the longitude of perihelion, these elements being those of any ellipse which approximates to the true orbit. Then, if we suppose that v, r are expressed in the forms

$$v = l + \varpi + \text{func. } e, l, \qquad r/a = 1 + \text{func. } e, l,$$

we have

$$\frac{\partial}{\partial l}\left(\frac{r^2}{\Delta}\right) = \frac{\partial r}{\partial l}\cdot\frac{\partial}{\partial r}\left(\frac{r^2}{\Delta}\right) + \frac{\partial v}{\partial l}\cdot\frac{\partial}{\partial v}\left(\frac{r^2}{\Delta}\right).$$

With the help of the relations,

$$\frac{\partial v}{\partial l} = \frac{a^2(1-e^2)^{\frac{1}{2}}}{r^2}, \qquad \frac{\partial r}{\partial l} = \frac{ae\sin f}{(1-e^2)^{\frac{1}{2}}}, \quad . \quad . \quad (11.2)$$

we deduce, for the three components of $1/\Delta$,

$$\frac{\partial}{\partial v}\left(\frac{1}{\Delta}\right) = \frac{1}{a^2(1-e^2)^{\frac{1}{2}}}\frac{\partial}{\partial l}\left(\frac{r^2}{r'\Delta_0}\right) - \frac{e\sin f}{ar(1-e^2)}\left\{\frac{3}{2}\frac{r^2}{r'\Delta_0} + \frac{1}{2}\frac{r^2}{r'}\frac{1-\rho^2}{\Delta_0^3}\right\}, \quad (11.3)$$

$$r\frac{\partial}{\partial r}\left(\frac{1}{\Delta}\right) = \frac{1}{r^2}\left\{-\frac{1}{2}\frac{r^2}{r'\Delta_0} + \frac{1}{2}\frac{r^2}{r'}\cdot\frac{1-\rho^2}{\Delta_0^3}\right\}, \quad . \quad . \quad . \quad . \quad (11.4)$$

$$\frac{\partial}{\partial z}\left(\frac{1}{\Delta}\right) = -\frac{z-z'}{r^2r'^2(1-\rho^2)}\left\{r^2 \cdot \frac{1-\rho^2}{\Delta_0^3}\right\}, \quad . \quad . \quad . \quad . \quad (11.5)$$

where z, z' are the distances of the planets from the plane of reference.

12. These results suggest the calculation of

$$\frac{r^2}{\Delta} = \frac{r^2}{r'\Delta_0}, \qquad \frac{r^2(r'^2 - r^2)}{\Delta^3} = \frac{r^2}{r'}\cdot\frac{1-\rho^2}{\Delta_0^3}, \quad . \quad . \quad (12.1)$$

* When $r > r'$, define Δ_0 by $\Delta^2 = r^2\Delta_0^2$ with $\varrho \equiv r'/r$, and with other obvious modifications of the formulæ which follow.

rather than of $1/\Delta$, $1/\Delta^3$ by harmonic analysis : the amount of calculation is about the same. After it is completed so that the two functions are expressed as cosines and sines of multiples of l, l' with numerical coefficients, we have to multiply two double series respectively by

$$\frac{a \sin f}{r}, \quad \frac{a^2}{r^2} \quad . \quad . \quad . \quad . \quad (12.2)$$

to obtain the second term of (11.3) and the whole of (11.4) ; the first term of (11.3) is obtained by forming a derivative with respect to l. The formation of (11.5) requires the product of the second function by the series for $(z - z')/r^2r'^2(1 - \rho^2)$.

The amount of calculation needed to form these products is much less than appears at first sight. The series (12.2) being single Fourier series in sines and cosines of l respectively, and proceeding along powers of e, contain very few terms : for most purposes not more than 5 will be needed. The second term of (11.3) contains the factor e and (11.4) is not needed to the degree of accuracy of (11.3), since (11.3) gives the principal parts of the coefficients of the long-period terms which determine the number of places of decimals to be used. The first factor of (11.5) is a rapidly converging double series : but it is needed with still lower accuracy, because it has the inclination as a factor and also because the solution of the corresponding differential equation involves only the first power of any small divisor ; it suffices, in fact, to choose out only those products which are needed.

The calculation of the " remainders " of the two functions (12.1) is made by means of the formulæ of § 6, with ρ replacing a, and the factors r^2/r', $r^2(1 - \rho^2)/r'$ for the respective cases $s = \frac{1}{2}, \frac{3}{2}$ inserted.

The angle ψ is calculated from the usual formulæ : it depends on the true longitudes, the longitudes of the nodes and the inclinations. All the elements are supposed to have received numerical values at the outset.

13. *Summary of the Calculation of the Special Values of the two Functions* (12.1) *and of the Expressions for the Remainders.*—The numerical values of the elements of the orbits of both planets are supposed to be given (the " epochs " are not needed and the mean motions are used only to get the mean distances). The variables which receive special values are the two mean anomalies.

Compute the special values of r/a and of $v - \varpi - l$ which depend only on e, l. Similarly those of r'/a', $v' - \varpi' - l'$, depending only on e', l'. Deduce the special values of $\rho = r/r'$, v, v', ψ.

Compute the functions (12.1). This may be done in the forms

$$\begin{aligned} \frac{r^2}{\Delta} &= a \cdot \frac{r}{a}\left\{\frac{\rho}{2(\lambda - \cos\psi)}\right\}^{\frac{1}{2}}, \\ \frac{r^2(r'^2 - r^2)}{\Delta^3} &= a \cdot \frac{r}{a}\left(\frac{1}{\rho^2} - 1\right)\left\{\frac{\rho}{2(\lambda - \cos\psi)}\right\}^{\frac{3}{2}}, \end{aligned} \quad (13.1)$$

where $\lambda = \frac{1}{2}(\rho^{-1} + \rho)$. The number of places of decimals in the second function may, in general, be one less than that used in the first function.

Compute the values of ϕ from

$$\cos\phi = \frac{\lambda^2 - 1}{\lambda - \cos\psi} - \lambda \quad \text{or} \quad \sin\phi = \frac{\frac{1}{2}\left(\frac{1}{\rho} - \rho\right)\sin\psi}{\lambda - \cos\psi}.$$

Compute the " remainders " for the functions (12.1). These special values are found by multiplying the functions (6.4) by $a \cdot r^2/r'a$, $a \cdot r^2(1 - \rho^2)/r'a$, respectively, and computing the special values of the two products.

After subtraction of the special values of the remainders from those of (12.1), each from each, the results are ready for harmonic analysis.

Harmonic Analysis.

14. When numerical harmonic analysis is used to expand a given oscillating function into a Fourier series, we give the angle a number of values, the same as the number of required coefficients, insert them in the function and the assumed series, and then solve the resulting linear equations so as to obtain the coefficients. Thus if

$$f(\theta) = a_0 + \Sigma a_i \cos i\theta, \quad i = 1, 2, \ldots$$

we decide in advance to stop at some value j of i, so that there are $j + 1$ coefficients a_i to be determined from the $j + 1$ equations

$$f(\theta_\kappa) = a_0 + \Sigma_1^j a_i \cos i\theta_\kappa, \quad \kappa = 0, 1, \ldots j,$$

where θ_κ are the chosen numerical values of θ.

In general, the theoretical series is infinite. The errors committed by stopping at the $(j + 1)$th terms depend on the values chosen for θ_κ. If, as is usually most convenient, we choose $\theta_\kappa = \kappa\pi/j$, the errors committed are :—

$$\text{in } a_0,\ a_{2j} + a_{4j} + \ldots; \qquad \text{in } a_j,\ a_{3j} + a_{5j} + \ldots;$$

$$\text{and, generally, in } a_{j'},\ a_{2j-j'} + a_{2j+j'} + a_{4j-j'} + a_{4j+j'} + \ldots, \ 0 < j' < j.$$

For sine-series we have $a_0 = 0$ and the coefficients $a_{\lambda j}$ (λ = an integer) are absent from all the equations, so that a_j is not determined and the errors of the remaining a_i are those shown above.

15. When we put

$$f(\theta) = a_0 + \Sigma_1^j (a_i \cos i\theta + a_i \sin i\theta),$$

it is convenient to use the values $\theta_\kappa = 2\kappa\pi/j$, and then

$$f(\theta) + f(2\pi - \theta) = 2a_0 + 2\Sigma_1^j a_i \cos i\theta,$$

$$f(\theta) - f(2\pi - \theta) = 2\Sigma_1^j b_i \sin i\theta.$$

Further, if an even number be chosen for the value of j, each set may be divided into two others, one containing only odd values of i and the other even values. If $j/2$ be odd, the values $\theta = 90°, 270°$ do not occur in the group and may be used to determine the missing coefficient b_j.

Where many such analyses have to be performed, the choice of the values of θ_κ is an important factor in reducing the work to a minimum.

In the applications of this paper, however, even values of $j/2$ seem to be called for, owing to the greater convenience in using angles whose sines and cosines are expressible by rational fractions. Of the values $j = 8, 12 \ldots$, the value $j = 16$ appears to be best adapted for securing sufficient accuracy with simplicity in the determination of the errors. This gives $\theta_\kappa = 22°{\cdot}5 \,.\, \kappa$. In some problems there may be advantage in the use of $\theta_\kappa = 30° \,.\, \kappa$ combined with $\theta_\kappa = 45° \,.\, \kappa$, especially when the latter are only needed for a portion of the work. For higher accuracy we can conveniently use $\theta_\kappa = 18°\kappa$ or $\theta_\kappa = 15°\kappa$. In all these cases the solution of the equations for finding the coefficients involves much less labour than the calculation of the special values of the function, if the work be well arranged.

In general, the errors with the use of $\theta_\kappa = 2\pi\kappa/j$ are those shown in § 13. Properties of the function can, however, be utilised to prove that certain of the actual errors are less than those given by the formula.

16. *Double Harmonic Analysis.*—The problem is the expansion of a function of two independent angles θ, θ' in cosines and sines of the multiples $i\theta \pm i'\theta'$. I suppose that for θ we use the j special values $2\pi\kappa/2j \equiv \kappa\alpha$ each combined with the j' special values $2\pi\kappa'/2j' \equiv \kappa'\alpha'$ of θ', where $\kappa = 0, 1, \ldots 2j-1$ and $\kappa' = 0, 1, \ldots 2j'-1$.

The analysis is made with the form

$$f(\theta, \theta') = \mathrm{A} + \mathrm{B} + \mathrm{C} + \mathrm{D},$$

where

$$\mathrm{A} \equiv \Sigma a_{i,i'} \cos il \cos i'l', \qquad \mathrm{C} \equiv \Sigma c_{i,i'} \sin il \cos i'l',$$
$$\mathrm{B} \equiv \Sigma b_{i,i'} \sin il \sin i'l', \qquad \mathrm{D} \equiv \Sigma d_{i,i'} \cos il \sin i'l',$$

in which $i = 0, 1, \ldots j$ and $i' = 0, 1, \ldots j'$.

After the calculation of the special values of f, the solution of the equations is performed in the following way. The four sets of terms are separated by the relations

$$\begin{array}{llllllllll}
+f(\kappa\alpha, & \kappa'\alpha') & +f(2j\alpha-\kappa\alpha, & 2j'\alpha'-\kappa'\alpha') & +f(\kappa\alpha, & 2j'\alpha-\kappa'\alpha') & +f(2j\alpha-\kappa\alpha, & \kappa'\alpha') & =4\mathrm{A}, \\
+ & ,, & + & ,, & - & ,, & - & ,, & =4\mathrm{B}, \\
+ & ,, & - & ,, & + & ,, & - & ,, & =4\mathrm{C}, \\
+ & ,, & - & ,, & - & ,, & + & ,, & =4\mathrm{D}.
\end{array}$$

Each of these sets is to be analysed separately with the values $\theta_\kappa = \kappa\alpha$, $\theta_{\kappa'} = \kappa'\alpha'$ as the special values.

17. Let the values of 4A be written in the form of an array with $j+1$ columns and $j'+1$ rows. Each column is analysed for the values of l as a cosine Fourier series and it then gives the values of the coefficients of $\cos il$, *i.e.*, it gives the values of $4\Sigma_{i'} a_{i,i'} \cos i'\theta'$ for each of the values of θ'.

The numbers in a row corresponding to a given value of i are then rewritten in a column, and each column is analysed as a cosine Fourier series for values of θ'. The headings of these columns are the multiples of θ and the coefficients finally found are the required $a_{i,i'}$.

A similar procedure is followed with B, C, D, except that we have to perform sine as well as cosine analyses. In the sine analyses, the values for $\theta = 0, \pi$ and $\theta' = 0, \pi$ are zero on both sides of the equation so that the coefficients $b_{j,j'}$, $b_{i,j'}$, $c_{j,i'}$, $d_{i,j'}$ are not found. The number of determined coefficients in the four sets is thus :

$$(j+1)(j'+1) + (j-1)(j'-1) + (j-1)(j'+1) + (j+1)(j'-1) = 4jj'.$$

18. *Application to the Disturbing Function.*—As stated earlier, the values $j = j' = 8$, with θ, θ' as the two mean anomalies, seem to be those which best fit the various phases of the problem for the majority of cases. (Very considerable gains in accuracy could be obtained in certain problems, *e.g.* in that of the mutual perturbations of Jupiter and Saturn, if we could use θ = the difference of the mean anomalies, θ' = the mean anomaly of Jupiter with $j = 12$, $j' = 8$, and remainders with $i = 9, 17$ successively ; but a more careful study of the arrangement of the work than that which I have given to this case should be made before calculation is started on this plan.)

With $j = j' = 8$ and the remainder formula of § 6 with $i = 9$, we are attempting to determine those coefficients for which $i<8$, $i'<8$ in the formulæ of § 15. But the results when put into the form

$$\Sigma\lambda_{i,i'} \cos(i\theta - i'\theta' + \mu_{i,i'}) + \Sigma\lambda'_{i,i'} \cos(i\theta + i'\theta' + \mu'_{i,i'}), \qquad (18.1)$$

show that the numerical work is unchanged if we put $i + 8$ for i with $i' + 8$ for i'. Further, if we put $16 - i$ for i or $16 - i'$ for i', the two sets of terms interchange, so that it is necessary to find the degree of accuracy obtained in any coefficient.

19. The properties of the disturbing function give additional information. Define the " order " of a coefficient as the sum of the powers of the eccentricities and inclination by which the coefficient is divisible. It is assumed that all coefficients of order greater than 8 are insensible, but this assumption with respect to those parts, in a coefficient of lower order, which are of order greater than 8 is not necessary. Then, remembering that the maximum multiple of $\theta - \theta'$ with a coefficient of zero order present in the functions we use is 8, we deduce the following :—

(i) The order of $\lambda_{i,i'}$ is $|i - i'|$ when $i, i' \gtreqless 8$ and is $i + i' - 16$ when $i, i' > 8$. The maximum value of $i + i'$ is 24.

(ii) The order of $\lambda'_{i,i'}$ is $i + i'$, so that, in this set of terms, $i + i' \leqq 8$.

With the help of these considerations we see :

(*a*) That coefficients $\lambda_{i,i'}$, $\lambda'_{i,i'}$ for which $i + i' < 8$ are determined to the 8th order at least, and that there are no other sensible coefficients amongst the $\lambda'_{i,i'}$ for any values of i, i'.

(*b*) That when $16 \gtreqless i + i' \gtreqless 8$, the error of $\lambda_{i,i'}$ is of order $16 - i - i'$. For these cases we have $b_{i,i'} = a_{i,i'}$, $d_{i,i'} = -c_{i,i'}$ (§ 18) except when $i + i' = 8$, when the respective differences are of the 8th order. The additional work necessary to find the coefficients b, d is so small that they should be computed ; they serve as tests of the accuracy of the work and of the degree of accuracy attained. If, instead of $2\pi/16$, we use multiples of $2\pi/2n$ for the special values, we replace 8 by n and 16 by $2n$ in the above argument.

No account has been taken of powers of ρ in the above statements. Their practical effect is in general to increase the accuracy of the coefficients in the cases (*b*).

20. In general, the coefficients not determined will not be needed. It is only for those values of i, i' in $il - i'l'$ with i or i' greater than 8 which give long-period arguments that the exceptions arise. Such terms are of high order and are troublesome to determine accurately by any known method. For example, in the determination of the mutual perturbations of Jupiter and Saturn, the method gives the coefficient of the argument $5S - 2J$ with sufficient accuracy, but it does not give that of $10S - 4J$, and it is doubtful whether any plan which involved the use of the remainder formula would be economical for this latter term: apparently the number of special values would have to be more than doubled—24×24 instead of 16×16, with, however, $i = 13$ in the remainder formula.

A word of caution may be added here for anyone intending to apply the formulæ of this paper to any particular case. While improvements can doubtless be suggested, much time has been spent on the consideration of the various details. Limitations of space forbid a full discussion of the reasons which have led to the adoption of the various forms and numbers chosen. The final choice of 16 values for l and 16 for l', for example, involved a careful examination of every part of the work, demanding an amount of time which seemed quite out of proportion with the rest of the work when the result was finally set down. Time and labour will ultimately be saved if, before changes in the plans outlined here or in the numbers recommended are made, a similar careful examination of the effects of such changes on each part of the work be undertaken. Numerical calculation seems to be an "unstable" process: a minor change in a formula will often greatly alter its amount, chiefly because the mechanical aids to the computer in the form of tables, machines, or other devices are limited.

21. The experience gained in the extensive use of double harmonic analysis on another problem may be summed up in the following statements which indicate some defects and some advantages:—(i) The amount of computation required is very dependent on the arrangement of the work; (ii) it is easily adapted for routine computation; (iii) in the application to the disturbing function, the work of analysis is much less than that required for the computation of the special values of the function; (iv) it is not necessary to calculate the special values of the function to more places than those actually needed in the coefficients; (v) all the special values must be taken to the same number of places of decimals; (vi) coefficients not needed or insensible are computed, the latter, however, serve as tests when theory indicates their existence; (vii) the choice of the special values of the two angles, both as to their number and magnitude, is of considerable importance; (viii) with proper choice and arrangement nearly all the work of analysis can be reduced to additions and subtractions of pairs of numbers.

Yale University:
1928 *May.*

Note on the Separation of Gases in Prominences. By S. R. Pike.

§ 1. Expressions have recently been worked out * for the velocities and accelerations of atoms in the sun's chromosphere on the assumption that the motions are controlled entirely by gravity and radiation-pressure; and an attempt was made to show that nearly all the observed phenomena could be accounted for in this way alone. Exact results, however, could only be given in one case, viz. that of Ca^+, owing to lack of the necessary data; but further measurements of the type required have lately become available,† not only for Ca^+ but also for several other elements. It therefore seemed worth while to work out in greater detail the numerical treatment of certain equations which had already been given; and the results are contained in the present note.

Their chief importance consists in indicating the possibility of a definite observational test of the radiation-pressure theory; for, as will be seen from the following, the theory predicts a considerable "sorting out" of the various constituents of a mixture of gases moving under radiation-pressure. No observations appear to have been made up to the present on the simultaneous motions of different gases in one and the same prominence; but the order of magnitude predicted by the theory is such that the effect, if present, should be easily observable.

§ 2. *Motion over a Sunspot.*—On account of the lack of radiation, gases near a sunspot will move downwards and horizontally towards it, until the increased intensity due to the Doppler effect counterbalances the "attraction" of the spot. In this way limiting velocities are attained, and a rough estimate of their magnitude has already been given.‡ We require a more detailed treatment of the equations for our present purpose, however.

The horizontal force H is given by

$$\left(\frac{4\pi}{mg_\odot}\right)\mathrm{H} = x\mathrm{J} + xp\left(\mathrm{J}' + \mathrm{J}''\frac{u}{v}\right) - pv\left(\frac{8}{3}\sin^{-1}\frac{u}{w} + \frac{3\pi}{4}\frac{u}{w}\right). \quad (1)$$

where

$$\mathrm{J} = 2\mathrm{J}_\beta + 3\mathrm{J}_\alpha, \qquad \mathrm{J}' = 2\mathrm{J}'_\beta + 3\mathrm{J}'_\alpha, \qquad \mathrm{J}'' = 2\mathrm{J}''_\beta + 3\mathrm{J}''_\alpha,$$

taking as usual,

$$\alpha = \tfrac{3}{5}, \qquad \beta = \tfrac{2}{5},$$

$$x = 0{\cdot}86,$$

u, v = vertical and horizontal velocities,

and

$$p = \frac{\nu_0}{c}\cdot\frac{\delta\mathrm{I}}{\mathrm{I}'\delta\nu}$$

in the previous notation.

* *M.N.*, **88**, 3.
† Unsöld, *Zeit. f. Phys.*, **46**, 765; Minnaert, *Zeit. f. Phys.*, **45**, 610.
‡ *Loc. cit.*, Tables XII. and XII*a*.

Similarly the vertical force is given by

$$\left(\frac{4\pi}{mg_\odot}\right)V = xI + xp\left(I' + I''\frac{u}{v}\right)v - pv\left[\frac{8}{3}\left(1 + \frac{u}{v}\sin^{-1}\frac{u}{v}\right) + \frac{3\pi}{4}\left(\frac{w}{v} + \frac{u^2}{wv}\right)\right] \quad (2)$$

where

$$I = 2I_\beta + 3I_a, \text{ etc.}$$

The quantities I, J, I′, J′, I″, J″ can all be found by means of the tables which have already been given. These have been calculated for an atom situated symmetrically with respect to a square sunspot of side $2a$, at a height na above it, and at a horizontal distance ma from its nearer edge ; and the values of the various functions are tabulated for

$$m = 0, 1, 2, 4, 6, 8.$$
$$n = 1, 2, 3, 4, 6.$$

The equations are written in the above form because it is convenient to take v and u/v as the independent variables.

The limiting velocities are found by writing H = 0, V = 0 ; so that we get, from (1),

$$v = \frac{xJ}{P} \cdot \frac{1}{\left(\frac{8}{3}\sin^{-1}\frac{u}{w} + \frac{3\pi}{4}\frac{u}{w}\right) - x\left(J' + J''\frac{u}{v}\right)} \quad . \quad . \quad (3)$$

and a similar expression from (2). Eliminating v between them we have at once

$$\frac{I}{J} = \frac{\left[\frac{8}{3}\left(1 + \frac{u}{v}\sin^{-1}\frac{u}{w}\right) + \frac{3\pi}{4}\left(\frac{w}{v} + \frac{u^2}{wv}\right)\right] - x\left(I' + I''\frac{u}{v}\right)}{\left[\frac{8}{3}\sin^{-1}\frac{u}{w} + \frac{3\pi}{4}\frac{u}{w}\right] - x\left(J' + J''\frac{u}{v}\right)} \quad . \quad (4)$$

This equation enables us to find u/v, and thence the value of v follows at once from equation (3).

It is important to note that u/v depends upon x, and upon (m, n), but not upon p, *i.e.* it does not involve the structure of the absorption line, and so will be approximately the same for all atoms at any given point.

The numerical treatment brings out the further fact that the R.H.S. of equation (4) is never less than about 2·5, so that a limiting velocity will only exist at points where

$$\frac{I}{J} > 2{\cdot}5.$$

At points where $I/J < 2{\cdot}5$ it is not possible to find any value of u/v which will satisfy (4), and consequently no limiting downward velocities can exist.* This is an important restriction, and implies that among

* At such points it appears that the horizontal motion towards the spot will be associated with an *upward* limiting velocity. This case is, however, not covered by the expressions which have been given.

the cases which have already been worked out, it is only for the following that limiting velocities can be found, viz. :—

$$m = 0, \qquad n = 2, 3, 4, 6.$$

At these points, it is possible to solve equation (4) numerically for u/v ; and it appears that there are two roots, one $>0{\cdot}75$ and the other $<0{\cdot}75$. The approximate values are given in Tables I. and II.

Table I.

Values of u/v ($<0{\cdot}75$).

$m=0$, $n=$	2	3	4	6
$u/v=$	0·64	0·38	0·25	0·225

Table II.

Values of u/v ($>0{\cdot}75$).

$m=0$, $n=$	2	3	4	6
$u/v=$	1·10	1·70	2·51	2·94

In practice, it is evident that the relevant values are those of Table II., viz. $u/v>1$. For we are dealing with points where $I/J>2$, *i.e.* where the initial vertical acceleration is more than twice as great as the horizontal acceleration, so that, except in special circumstances, we are likely to have $u>v$.

The next step is to find the values of v, and so of u, at these different points by means of equation (3) ; and in order to do so, we require the value of p ; *i.e.* we want to know the residual intensity and the gradient of the absorption line.

The following figures have been deduced from the curves given by Unsöld :

Table III.

Element.	Line.	I'.	p.		p (Schwarzschild).	p (Minnaert).
Ca^+	H	0·16	$7{\cdot}7\times10^{-3}$	$6{\cdot}6\times10^{-3}$	$6{\cdot}0\times10^{-3}$	
	K	0·16	$5{\cdot}5\times10^{-3}$		$1{\cdot}06\times10^{-2}$	$1{\cdot}62\times10^{-2}$
Sr^+	λ4215	0·31	$9{\cdot}5\times10^{-2}$	$9{\cdot}8\times10^{-2}$		
	λ4078	0·25	10×10^{-2}			
Ba^+	λ4934	0·50	6×10^{-2}	6×10^{-2}		
	λ4554	0·46	6×10^{-2}			

The last two columns contain the results of measurements by Schwarzschild and Minnaert ; but since for comparative purposes it is important that the data should be homogeneous, we employ only Unsöld's figures in the following calculations.

The limiting velocities for Ca^+ can be found at once from equation (3), and are as follows :—

TABLE IV.

Limiting Velocities for Ca^+ in km./sec.

$m=0$, $n=$	2	3	4	6
u (vertical)	53	25	16	6·0
v (horizontal)	48	15	6·3	2·05

whilst for Sr^+ we find :

TABLE V.

Limiting Velocities for Sr^+ in km./sec.

$m=0$, $n=$	2	3	4	6
u (vertical)	3·6	1·7	1·1	0·41
v (horizontal)	3·25	1·0	0·42	0·14

(For atoms of Ba^+ the limiting velocities are about 60 per cent. larger than those given in Table V. ; but it seems doubtful whether the theory applies here. The barium lines extend to such a small height in the chromosphere that the Ba^+ is probably not fully supported by radiation-pressure.)

These, then, are the maximum velocities with which we should expect the calcium and strontium atoms in various positions to be moving towards the sunspot ; and it is evident that there should be a very marked " sorting out " effect. From the physical point of view, this is to be ascribed simply to the great breadth of the H, K lines, and the narrowness of the lines due to Sr^+. The former have, in fact, a half-width of about 10 A.U., corresponding to a " velocity of escape " of the order 700 km./sec. ; whilst the half-width of the strontium lines is about 0·5 A.U., and the velocity of escape 40 km./sec. or so. Thus if two atoms, one Ca^+ and the other Sr^+, start together from rest, they will have equal initial accelerations, but the Sr^+ atom has only to acquire a relatively small velocity before it begins to absorb in a region of increased intensity, and so to experience an increasing head-resistance. It therefore reaches a limiting velocity which is very much smaller than that of the Ca^+ atom. The exact expression of this principle is contained in Tables IV. and V.

With regard to the smallest velocities in the tables, it does not seem likely that there is any physical significance to be attributed to the precise values given ; for a certain minimum velocity is necessary to overcome the bluntness of the absorption-line contour. For Ca^+ 4 or 5 km./sec. should be sufficient ; whilst for Sr^+ the whole line is so much narrower that a velocity of $\frac{1}{2}$ km./sec. or even less would probably be enough for the purpose.

§ 3. *Motion above a Facula.*—Now let us turn to the consideration of the converse case, viz. the motions of gases in ordinary eruptive prominences. These are supposed to be due to the development of faculæ, and the velocities will therefore be upwards and away from the disturbed areas ; and, further, must always be accelerated.* Moreover,

* Limiting velocities of about the same order for all elements will occur at very great distances from the sun (Milne, *M.N.*, **86**, 465).

the acceleration of a given atom will depend, not only on its position and on the temperature of the facula, but also, in general, upon the wave-length and structure of its principal absorption line, and thus (on account of the Doppler effect) will be a function of its velocity. If, therefore, an eruptive prominence is really due to radiation-pressure, it is possible that its various constituents should separate out as it rises; and by a simple extension of formulæ already given we can estimate the order of magnitude to be expected for this effect.

The simplest case to take is that of an atom situated on the axis of a circular disc, so that there is only a vertical motion to deal with. If f is the acceleration expressed in terms of $g_\odot$, and a, na are the radius of the disc and the height of the atom above it respectively, then we find at once.

$$f = \frac{x}{\beta + \frac{2}{3}\alpha}\left[\beta \cdot \frac{1}{1+n^2} + \frac{2}{3}\alpha\left(1 - \frac{n^3}{(1+n^2)^{3/2}}\right)\right] + \frac{up}{\beta + \frac{2}{3}\alpha}\left[\left(\frac{2}{3}\beta + \frac{1}{2}\alpha\right) + x\left\{\frac{2}{3}\beta\left(1 - \frac{n^3}{(1+n^2)^{3/2}}\right) + \frac{1}{2}\alpha \cdot \frac{1+2n^2}{(1+n^2)^2}\right\}\right] \quad (5)$$

in the previous notation; or

$$f = A + Bp \cdot u \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where A, B depend only on the position of the atom and on the wave-length of its principal line.

If we take the temperature of the facula as 7500° A., then

$$x = 2{\cdot}4 \text{ at } \lambda = 4000 \text{ A.U.},$$

and we can use the known values of p for Ca^+ and Sr^+ to find the accelerations of atoms of these substances in any required circumstances. For example, the vertical accelerations corresponding to different vertical velocities at $n = 1$, $n = 2$, and $n = 5$ have been calculated, and are given in Table VI.

TABLE VI.

Vertical Accelerations Over a Circular Facula.

	$u=$	0	1	5	10	20	50	km./sec.
$n=1$	Ca^+	1·36	1·37	1·42	1·48	1·61	1·99	
	Sr^+	1·36	1·54	2·13	2·91	4·46	9·11	
	$u=$	0	1	5	10	20	50	km./sec.
$n=2$	Ca^+	0·57	0·58	0·61	0·65	0·73	0·95	
	Sr^+	0·57	0·68	1·13	1·68	2·79	6·12	
	$u=$	0	1	5	10	20	50	km./sec.
$n=5$	Ca^+	0·11	0·12	0·14	0·17	0·22	0·39	
	Sr^+	0·11	0·19	0·50	0·92	1·70	4·0	

Accelerations expressed in terms of $g_\odot$.

It will be seen at once that as soon as an appreciable upward velocity has been acquired, the strontium atoms will be subjected to a much larger acceleration than the calcium atoms; and so the two gases should be quickly sorted out in an eruptive prominence in which they start together. This is similar to the effect which we have already seen is to be expected near a sunspot, but it is in the opposite sense; for there the strontium acquires a lower limiting velocity than the calcium, whereas in a prominence it should be moving faster. This may be illustrated most clearly by taking the case which is the exact converse, viz. an atom on the axis of a circular spot. Its limiting downward velocity can be found at once from equation (6), putting

$$x = -0{\cdot}86$$

and

$$u = -A/Bp \quad . \quad . \quad . \quad . \quad (7)$$

Numerical results are given in Table VII.

TABLE VII.

Limiting Vertical Velocities Over Circular Sunspots.

n.	Ca+	Sr+
	km./sec.	
0·5	800	55
1·0	260	18
1·5	103	7·2
2·0	55	3·8
3·0	26	1·8
5·0	9	0·6

§ 4. Similar considerations can be applied in a general way to other elements which occur in prominences, such as H, Fe, Ti, etc.; though, as these atoms are not of the simple Ca^+ type, their motions cannot be submitted to precise calculations. It is evident, however, that the main controlling factor is the width of the absorption lines; and not, as is often supposed, the atomic weight. The phenomena are, in fact, independent of the latter quantity.

Thus the Balmer lines of hydrogen, as may be seen from Unsöld's curves, are broad and have flat minima; so that changes in intensity due to the Doppler effect will not become appreciable until high velocities are attained. We should therefore expect H and Ca^+ to be associated together in prominences. Fe and Ti, on the other hand, have much narrower absorption lines, and they should, therefore, be associated with Sr^+. Between the two groups of elements, however, there must be a clearly marked separation; above a sunspot, the inflow of H and Ca^+ should be very much faster than that of Sr^+, Fe, and Ti; whilst in an eruptive prominence, the latter group should move with a larger acceleration than the former, and the two will then rapidly separate out.

The University, Leeds:
April 1928.

On the Advance of Periastron in Eclipsing Binaries.

By Henry Norris Russell.

It is well known that if a small satellite moves about an oblate planet in an orbit of small eccentricity with a period P, the periastron will advance with a period P′ such that

$$P/P' = \frac{r^2}{a^2}(e - \tfrac{1}{2}\phi),$$

where a is the radius of the orbit, r that of the planet, e the superficial ellipticity of the latter, and ϕ the ratio of centrifugal force to gravity at its equator. The value of $e - \frac{1}{2}\phi$ depends on the internal constitution of the planet, and ranges from $\frac{3}{4}\phi$ for a homogeneous body to zero for one in which all the mass is concentrated at the centre.

For two oblate spheroids in orbital motion about one another, the motion of periastron will be given, to the first order of small quantities, by

$$P/P' = \frac{1}{a^2}\{r^2{}_1(e_1 - \tfrac{1}{2}\phi_1) + r^2{}_2(e_2 - \tfrac{1}{2}\phi_2)\},$$

or, setting $e - \frac{1}{2}\phi = k\phi$,

$$P/P' = \frac{1}{a^2}(k_1 r^2{}_1\phi_1 + k_2 r^2{}_2\phi_2).$$

In eclipsing binaries there is usually direct evidence from the radiation effect that the periods of revolution and rotation of the stars coincide (at least on the surface). If m_1 and m_2 are the masses,

$$\phi_1 = \frac{m_1 + m_2}{m_1}\frac{r_1{}^3}{a^3},$$

and we have

$$P/P' = \frac{m_1 + m_2}{a^5}\left\{k_1\frac{r_1{}^5}{m_1} + k_2\frac{r_2{}^5}{m_2}\right\}.$$

If the ellipticity of the stars was due to their rotation alone, this equation would give the rate of the advance of periastron; but an additional ellipticity arises from the tidal distortion of each star by the other. The upward tidal force at the point on the equator of star 1 nearest to star 2 is $\frac{2m_2}{m_1 + m_2}$ times the centrifugal force at this point to the first order, while the downward force at all points 90° from this is half as great. To this order, the effect of this distortion will be independent of the other, and in the same proportion to the disturbing force, so that the difference between the longest diameter and the mean of the other two, due to tidal distortion, will be $\frac{3m_2}{m_1 + m_2}$ times that of the equatorial and polar diameters arising from the rotation. The perturbative effect is proportional to the difference between the diameter lying in the line of

centres and the mean of the other two, so that the tidal effect in star 1 is $\frac{6m_2}{m_1 + m_2}$ times the rotational. We find, then, for the whole rate of advance of periastron,

$$P/P' = k_1 \frac{r_1^5}{a^5}\left(1 + 7\frac{m_2}{m_1}\right) + k_2 \frac{r_2^5}{a^5}\left(1 + 7\frac{m_1}{m_2}\right).$$

The only eclipsing binary for which an advance of periastron appears to be definitely indicated is Y Cygni, for which Dugan and Waterfield (*Harvard Bulletin*, 856) find $P' = 5400P$. The two stars in this system are substantially equal in mass, and, according to Shapley (*Princeton Contributions*, No. 3, 89, 1915), they are equal in size, $\frac{r}{a} = 0{\cdot}167$. Assuming that the k's are equal, we find $k = 0{\cdot}076$.

For a homogeneous body $k = 0{\cdot}75$; for the Earth it is 0·47, for Jupiter 0·26, and for Saturn 0·12. It appears, therefore, that these stars must be even more condensed toward the centre than Saturn.

It is of much interest to compare this result with the theoretical values for stars of different constitution. The ellipticity of the surfaces of equal density within a mass of fluid rotating with uniform angular velocity, and with any assigned law of variation of density from the centre to the surface, may be compiled by equations due to Radau (Tisserand, *Traité de Mécanique Céleste*, tome II., p. 221, Paris, 1891).

For any such body

$$k = \frac{e - \frac{1}{2}\phi}{\phi} = \frac{3 - \eta_1}{2\eta_1 + 4},$$

where η_1 is the value at the surface of a quantity which is defined by the differential equation

$$a\frac{d\eta}{da} + \eta^2 - \eta - 6 + \frac{6\rho}{D}(\eta + 1) = 0.$$

Here a is the distance from the centre, on such a scale that $a = 1$ at the surface, D is the mean density and ρ the density at any point, and the integration starts with $a = 0, \eta = 0$.

Quadratures for the determination of η were made by the writer several years ago in connection with a study of stars built on Eddington's early model, in which the ratio β of gas pressure to total pressure was constant, and the equation of state of the gas was taken to be $P = \frac{R}{m}\rho T\frac{\rho_0}{\rho_0 - \rho}$, where ρ_0 is the limiting density attainable under very great pressure. When the inapplicability of this model became evident these computations were laid aside; now at last they may serve some useful purpose. For different values of the ratio of the central density ρ_c to ρ_0 they give spheres of different degrees of central condensation, which may be roughly indicated by the value of ρ_c/ρ_m, the ratio of the central to the mean density.

The quadratures gave the following results:

Table I.

$\frac{\rho_c}{\rho_0}$.	$\frac{\rho_c}{\rho_m}$.	k.	$\frac{\rho_c}{\rho_0}$.	$\frac{\rho_c}{\rho_m}$.	k.
0·00	54·2	0·0144	0·60	5·61	0·091
0·10	39·8	0·0175	0·70	3·65	0·147
0·20	27·5	0·0220	0·80	2·37	0·248
0·30	19·0	0·0290	0·90	1·53	0·430
0·40	12·8	0·0401	1·00	1·00	0·750
0·50	8·55	0·0586			

The first line of the table corresponds to a perfectly gaseous star on Eddington's model, the last to a homogeneous sphere.

The value derived from observation for Y Cygni corresponds approximately to $\rho_c = 6\rho_m$. This is a surprisingly small degree of central condensation. Other models (for example, Emden's polytropes) would give a different value of ρ_c/ρ_m for this value of k, but, in all probability, one of the same order of magnitude.

For stars built on Eddington's model, the rate of advance of the apse should be only one-fifth of the observed value.

For Y Cygni the spectral class is O9, and the mean density 0·17 that of the Sun. It appears improbable that the central condensation is as small as these calculations indicate. An explanation may be found in Jeans's proof (*M.N.*, **86**, 456, 1926) that, owing to "radiative viscosity" the inner portion of a star should rotate more rapidly than the outer layers. Except near the centre, the angular velocity is inversely proportional to the square of the radius, and in the central regions it may be ten times as great as at the surface.

An exact allowance for this effect would be difficult. To assume that the whole star rotates as fast as the inner layers will evidently exaggerate it; it may be concluded roughly that on Jeans's hypothesis the effect of rotational ellipticity may be ten times or even fifty times as great as has previously been calculated.

The tidal effect, which on the previous hypothesis amounts to three-quarters of the whole, will be unaltered, so that the rate of advance of periastron comes out from three to thirteen times that previously calculated.

The actual central condensations of Y Cygni should on this hypothesis be comparable with that of a uniformly rotating star for which k lies between 0·025 and 0·006, but is not likely to be much smaller. The values of ρ_c/ρ_m in Table I. are roughly proportional to those of $1/k$. On Jeans's theory, then, the central density in Y Cygni may be from 25 to 120 times the mean density, or from 6 to 30 times that of water. These are high values, but much lower than those suggested in a recent paper by Jeans (*M.N.*, **87**, 43, 1926), which are as high as 1000 for Sirius and 5000 for V Puppis (a star very similar to Y Cygni). With such high densities as these, most of the mass would be concentrated in a small volume near the centre, and even though the internal rotation were rapid the motion of the periastron would be slow.

The Orbits of Four Long-period Spectroscopic Binaries. By H. Spencer Jones, M.A., Sc.D., H.M. Astronomer.

Radial velocity observations with the 24-inch Victoria telescope of the Cape Observatory, employed in conjunction with the four-prism spectrograph, were terminated in 1925 April, in order that the telescope might be used for a programme of stellar parallax determinations. The results of the observations, not hitherto published in the *Cape Annals*, have been prepared for press and are now in course of publication as *Cape Annals*, vol. x., part 8. In preparing the material for press, orbital elements were derived for all the spectroscopic binaries—thirteen in number—for which sufficient material was available for the purpose.

Whilst these results were in the hands of the printer, vol. xvi. of the *Publications of the Lick Observatory* was published. This volume contains the results of the radial velocity determinations at the Lick Observatory and at the Santiago Station, from the commencement of the observations down to 1927 January 10. The catalogue of radial velocities include all stars brighter than visual magnitude 5·51 in the whole sky, with a few exceptions, together with a number of stars fainter than this limit—a total of 2771 stars. Comparing the Cape observations with the Lick and Santiago observations contained in this volume, it was found that there were a few stars for which the combined observations at Lick and at the Cape were sufficiently numerous and well distributed throughout the period of radial velocity variation to enable satisfactory orbital elements to be derived. The orbital elements of these stars are derived in the present paper, which completes part 8, vol. x. of the *Cape Annals*, in the sense that all Cape observations of stars with variable radial velocities which can be utilised at present for the determination of the spectroscopic orbits have now been utilised.

The four stars under consideration are all of long period, the periods ranging from 1066 to 4198 days. The number of spectroscopic binary systems with periods exceeding 1000 days for which orbital elements have been derived is very small. The stars coming under this category, which are listed in the "Third Catalogue of Spectroscopic Binary Stars," * are only six in number: α Phœnicis, period 3848 days; ϵ Hydrae, period 5588 days; β Capricorni, period 1375 days; o^2 Cygni, period 1170 days; ν Geminorum, period 9·6 years; ϵ Urs. Maj., period 4·15 years. γ Geminorum is excluded, as the elements are uncertain and do not represent later observations. The two giant stars α Orionis and α Scorpii are excluded, as they cannot be considered as binary systems. Of the six stars referred to, ϵ Hydræ is a close visual binary and is not therefore, strictly speaking, a spectroscopic binary. To these stars of long period may be added ζ Ceti, period 1652 days, and ξ Pavonis, period 2214 days, whose orbital elements I have derived in *Cape Annals*, vol. x., part 8, and the four stars whose orbits are derived in the present paper. One of these latter stars, α Tucanæ with period 4198 days, is

* *L.O.B.*, **11**, No. 355, 1924.

the longest-period spectroscopic binary known, leaving out of consideration visual binaries.

It is important for statistical purposes that orbits of as many long-period binaries as possible should be derived. These stars are apt to be overlooked in observing, as the periods of spectroscopic binaries are correlated with the velocity ranges according to a relation of the form

$$K = CP^{-\frac{1}{3}},$$

where K is the semi-amplitude of the velocity range, P is the period, and C is approximately constant. Long-period stars have therefore in general a small velocity range. It is possible that several observations of the star may be secured and the small velocity variations concealed by the accidental errors of observation, with the result that it may be concluded that the star has constant radial velocity and no further observations secured. It is probable that many stars of type G, K, or M whose velocities are regarded as constant are in reality spectroscopic binary systems with long period and small velocity range.

β Reticuli.

Boss 875 α $3^h\ 42^m{\cdot}9$ $\delta - 65°\ 7'$ (1900) $3^m{\cdot}80$ Ko.

The variable radial velocity of β Reticuli was announced by Palmer * in 1906, from early observations secured at the D.O. Mills Station of the Lick Observatory. The Lick observations secured to date comprise 18 plates obtained during the years 1903 to 1919. The Cape observations comprise 18 plates obtained during the years 1909 to 1925. Thirty-six plates in all are therefore available for the derivation of orbital elements, extending over the period 1903 to 1925.

The observations are summarised in Table I. The first column gives the plate number, the Lick plates being denoted by "L." The second

TABLE I.

Observations of β Reticuli.

Plate No.	Date.	J.D. 2,400,000+.	Revn. No.	Final Phase.	Observed Velocity.	Computed Velocity.	δV (O−C).
				d.	km./sec.	km./sec.	km./sec.
L	1903 Dec. 7	16456·7	−2	193·6	+54·2	+54·1	+0·1
,,	1904 Jan. 4	484·6	,,	221·5	53·4	53·5	−0·1
,,	Dec. 18	833·7	,,	570·6	48·0	48·1	−0·1
,,	1905 Nov. 5	17155·7	,,	892·6	47·1	47·0	+0·1
,,	Dec. 28	208·7	,,	945·6	47·4	47·1	+0·3
,,	1906 Oct. 26	510·9	,,	1247·8	47·6	49·0	−1·4
,,	Nov. 10	525·8	,,	1262·7	48·0	49·1	−0·6
,,	Dec. 26	571·7	,,	1308·6	50·5	49·6	+0·9
,,	1907 Oct. 24	873·8	,,	1610·7	52·9	54·3	−1·4
,,	Nov. 12	892·8	,,	1629·7	53·5	54·6	−1·1
2541	1909 Oct. 20	18600·6	−1	426·0	+47·5	+49·7	−2·2

* *L.O.B.*, **4**, 97, 1906.

TABLE I.—*continued.*

Plate No.	Date.	J.D. 2,400,000+.	Revn. No.	Final Phase.	Observed Velocity.	Computed Velocity.	δV (O−C).
				d.	km./sec.	km./sec.	km./sec.
2816	1910 Sept. 13	18926·6	−1	752·0	+49·6	+47·1	+2·5
3338	1911 Oct. 31	19341·6	,,	1167·0	47·1	48·2	−1·1
3360	Nov. 17	358·5	,,	1183·9	51·4	48·4	+3·0
3364	,, 29	370·4	,,	1195·8	53·7	48·5	[+5·2]
3804	1912 Sept. 10	656·7	,,	1482·1	52·8	52·1	+0·7
L	1913 Nov. 24	20096·6	0	10·5	57·6	57·1	+0·5
,,	1915 Sept. 16	757·9	,,	671·8	46·5	47·4	−0·9
,,	,, 28	769·7	,,	683·6	42·6	47·3	[−4·7]
,,	1916 Oct. 4	21141·7	,,	1055·6	47·4	47·5	−0·1
,,	,, 8	145·8	,,	1059·7	45·5	47·5	−2·0
,,	1917 Aug. 27	468·9	,,	1382·8	50·1	50·6	−0·5
,,	Sept. 19	491·7	,,	1405·6	48·0	50·9	−2·9
,,	1919 Jan. 29	988·6	,,	1902·5	56·9	57·3	−0·4
6434	1922 Nov. 10	23369·5	1	1371·9	50·2	50·4	−0·2
6436	,, 14	373·5	,,	1375·9	51·7	50·5	+1·2
6456	Dec. 9	398·5	,,	1400·9	51·9	50·8	+1·1
6457	,, 14	403·4	,,	1405·9	51·5	50·9	+0·6
6462	,, 20	409·4	,,	1411·9	52·9	51·0	+1·9
6766	1923 ,, 22	776·4	,,	1778·8	58·4	56·9	+1·5
6772	,, 29	783·4	,,	1785·8	56·6	57·0	−0·4
6793	1924 Jan. 17	802·4	,,	1804·8	57·6	57·1	+0·5
6798	,, 22	807·4	,,	1809·8	58·2	57·2	+1·0
7134	Dec. 11	24131·4	2	223·3	53·3	53·5	−0·2
7136	,, 13	133·4	,,	224·3	53·3	53·5	−0·2
7156	1925 Jan. 14	165·4	,,	256·3	+53·3	+52·8	+0·5

column gives the date of observation; the third the corresponding Julian Day to 0·1 of a day. The fourth and fifth columns contain respectively the revolution number and the phase in days computed from the final elements. The sixth column gives the observed velocity and the seventh column the velocity computed from the final elements. The last column contains the velocity residuals. Two large residuals are bracketed; these discordant plates were excluded from the least-squares solution.

The following preliminary elements were derived by the use of Russell's method:—*

$$\begin{aligned} P &= 1895 \text{ days} \\ G &= +52{\cdot}20 \text{ km./sec.} \\ K &= 5{\cdot}40 \text{ km./sec.} \\ e &= 0{\cdot}22 \\ \omega &= 354^\circ \\ T &= \text{J.D. } 2{,}419{,}984{\cdot}0. \end{aligned}$$

* Russell, *Ap. J.*, **40**, 282, 1914.

The notation is as in *Cape Annals*, vol. x., part 8, p. 32. K denotes the semi-amplitude of the velocity range, and the quantity G determines V_0, the velocity of the centre of mass of the system by means of the relationship

$$G = V_0 + Ke \cos \omega.$$

Normal equations were formed, based on the preliminary elements, by the use of Schlesinger's tables,* the period being retained in the least-squares solutions as an element requiring correction. The normal equations are :

δG	δK	π	ϵ	τ	m	
+ 34·00	− 5·05	− 5·37	+ 16·78	− 3·05	− 0·99	= + 2·80
	+ 15·46	+ 3·89	− 2·39	+ 2·41	+ 0·72	= − 3·42
		+ 18·54	− 4·02	+ 11·83	+ 2·29	= − 0·34
			+ 12·43	− 2·47	− 1·01	= + 1·41
				+ 7·78	+ 1·34	= + 0·42
					+ 4·19	= + 0·85

The quantities π, ϵ, τ, m are related to the differential corrections to the elements ω, e, T, P respectively, as follows :—

$$\pi = - K \,.\, \delta\omega$$
$$\epsilon = - K \,.\, 2{\cdot}21(1 - e^2)^{-1}\delta e$$
$$\tau = K \,.\, 2\pi(1 + e)^{\frac{1}{2}}(1 - e)^{-\frac{3}{2}}P^{-1}\delta T$$
$$m = K \,.\, 2\pi(1 + e)^{\frac{1}{2}}(1 - e)^{-\frac{3}{2}}P^{-2}\delta T.$$

The unit of time used in the coefficients of m was 4000 days.

The solution of the normal equations gave :

$$\delta G = - 0{\cdot}03 \pm 0{\cdot}28$$
$$\delta K = - 0{\cdot}21 \pm 0{\cdot}25$$
$$\pi = - 1{\cdot}87 \pm 1{\cdot}35 \qquad \delta\omega = + 19{\cdot}8^\circ \pm 14{\cdot}3^\circ$$
$$\epsilon = + 0{\cdot}12 \pm 0{\cdot}55 \qquad \delta e = - 0{\cdot}010 \pm 0{\cdot}044$$
$$\tau = + 2{\cdot}93 \pm 1{\cdot}96 \qquad \delta T = + 102{\cdot}1 \pm 68{\cdot}3$$
$$m = + 0{\cdot}34 \pm 0{\cdot}57 \qquad \delta P = + 16{\cdot}5 \pm 32{\cdot}2.$$

The final elements are therefore :

$$P = 1911{\cdot}5 \pm 32{\cdot}2 \text{ days}$$
$$G = + 52{\cdot}17 \pm 0{\cdot}28 \text{ km./sec.}$$
$$K = 5{\cdot}19 \pm 0{\cdot}25 \text{ km./sec.}$$
$$e = 0{\cdot}210 \pm 0{\cdot}044$$
$$\omega = 13{\cdot}^\circ 8 \pm 14{\cdot}^\circ 3$$
$$T = \text{J.D. } 2{,}420{,}086{\cdot}1 \pm 68{\cdot}^{d}3.$$

The probable errors have been derived from the final velocity

* *Allegheny Publications*, **1**, No. 6.

residuals, the probable error corresponding to unit weight (a single plate) being ± 0·92 km./sec.

From these elements may be derived V_0, the velocity of the centre of mass of the system, $a \sin i$, the projected semi-axis major and the mass function, $f = m_1{}^3 \sin{}^3 i/(m + m_1)^2$, in which m denotes the mass of the component whose spectrum is visible and m_1 that of the other component.

$$V_0 = + 51{\cdot}11 \text{ km./sec.}$$
$$a \sin i = 133{\cdot}4 \times 10^6 \text{ kms.}$$
$$f = 0{\cdot}02595\ \odot.$$

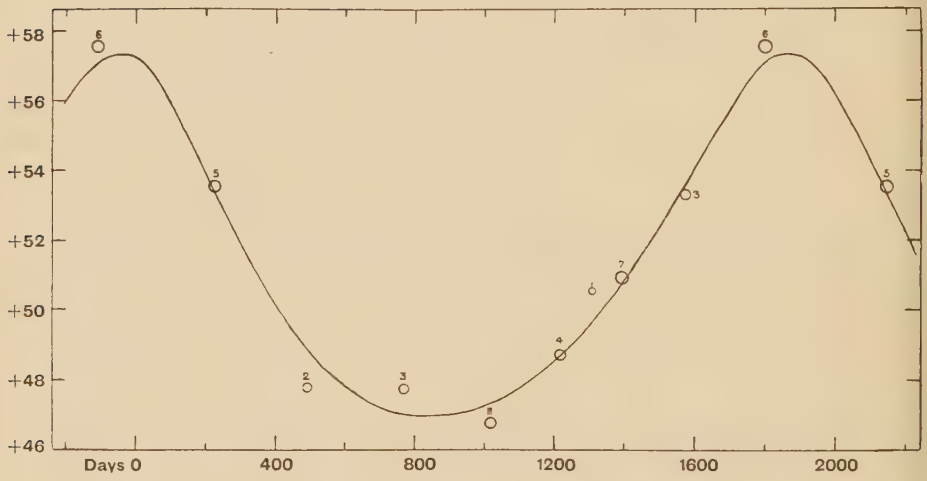


FIG. 1.—The Radial Velocity Curve of β Reticuli.

The radial velocity curve is shown in fig. 1, together with the observed velocities. Observations near the same phase have been grouped together, the number of plates combined being indicated in the diagram.

τ Puppis.

Boss 1772 α 6h 47m·5 $\delta - 50° 30'$ (1900) 2m·83 Ko.

The variable radial velocity of τ Puppis was announced by Curtis and Palmer * in 1908, from observations secured in the preceding years at the D.O. Mills Station of the Lick Observatory. The star was placed on the observing programme at the Cape in that year. The observations available for the derivation of the orbital elements comprise 26 Lick plates obtained between the years 1903 and 1920, and 31 Cape plates obtained between the years 1908 and 1924, giving a total of 57 plates.

The observations are summarised in Table II., in which the arrangement of the columns is the same as in Table I.

* *L.O.B.*, **5**, 60, 1908.

TABLE II.

Observations of τ Puppis.

Plate No.	Date.	J.D. 2,400,000+.	Revn. No.	Final Phase.	Observed Velocity.	Computed Velocity.	δV (O−C).
				d.	km./sec	km./sec.	km./sec.
L	1903 Nov. 15	16434·7	−5	771·9	+39·2	+39·4	−0·2
,,	1904 Oct. 12	766·8	−4	38·0	34·4*	37·3	−2·9
,,	Dec. 9	824·8	,,	96·0	33·4*	35·6	−2·2
,,	1905 Jan. 24	870·7	,,	141·9	32·8	34·4	−1·6
,,	Nov. 10	17160·8	,,	432·0	33·9	33·1	+0·8
,,	1907 Jan. 15	591·6	,,	862·8	39·5	40·5	−1·0
,,	,, 19	595·6	,,	866·8	40·1	40·5	−0·4
,,	Oct. 4	853·8	−3	59·0	36·4	36·7	−0·3
,,	,, 13	862·8	,,	68·0	37·0	36·4	+0·6
,,	Nov. 23	903·7	,,	108·9	35·2	35·3	−0·1
,,	1908 Jan. 11	952·7	,,	157·9	34·5	34·1	+0·4
2015	Nov. 16	18262·6	,,	467·8	34·3	33·6	+0·7
2231	1909 Feb. 16	354·4	,,	559·6	32·8	35·2	−2·4
2263	Mar. 3	369·4	,,	574·6	29·7	35·5	[−5·8]
2953	1911 Jan. 18	19055·5	−2	194·7	32·0	33·4	−1·4
3018	Feb. 21	089·3	,,	228·5	34·5	32·9	+1·6
3326	Oct. 26	336·6	,,	475·3	35·6	33·7	+1·9
3865	1912 ,, 23	699·7	,,	838·4	42·3	40·3	+2·0
L	1913 Dec. 1	20103·8	−1	177·0	35·8	33·7	+2·1
4256	1914 Mar. 14	206·3	,,	279·5	33·1	32·5	+0·6
L	1915 Sept. 22	763·9	,,	837·1	39·6	40·3	−0·7
,,	Oct. 18	789·9	,,	863·1	39·5	40·5	−1·0
,,	1916 ,, 18	21155·8	0	163·0	32·8	34·0	−1·2
,,	Nov. 14	182·9	,,	190·1	36·8	33·4	+3·4
,,	1917 Mar. 8	296·5	,,	303·7	30·5	32·4	−1·9
,,	,, 27	315·5	,,	322·7	29·8	32·4	−2·6
,,	Oct. 22	524·9	,,	532·1	35·7	34·7	+1·0
,,	,, 26	528·9	,,	536·1	34·7	34·7	0·0
,,	1918 Jan. 13	607·6	,,	614·8	35·9	36·3	−0·4
5173	Feb. 11	636·3	,,	643·5	37·5	36·9	+0·6
5180	,, 21	646·3	,,	653·5	36·0	37·1	−1·1
5183	,, 26	651·3	,,	658·5	36·3	37·2	−0·9
5185	,, 28	653·3	,,	660·5	38·1	37·3	+0·8
5189	Mar. 7	660·3	,,	667·5	37·9	37·4	+0·5
5193	,, 11	664·3	,,	671·5	+38·6	+37·5	+1·1

* Half weight (*L.O. Publ.*, **16**, 100, 1928).

TABLE II.—*continued.*

Plate No.	Date.	J.D. 2,400,000+.	Revn. No.	Final Phase.	Observed Velocity.	Computed Velocity.	δV (O−C).
				d.	km./sec.	km./sec.	km./sec.
5196	1918 Mar. 18	21671·3	0	678·5	+39·5	+37·7	+1·8
5199	,, 21	674·3	,,	681·5	35·4	37·7	−2·3
5206	Apr. 2	686·3	,,	693·5	36·2	37·9	−1·7
L	Dec. 16	944·6	,,	951·8	41·0	40·4	+0·6
,,	1919 Jan. 16	975·6	,,	982·8	40·1	40·1	0·0
,,	Mar. 20	22038·5	,,	1045·7	39·5	38·8	+0·7
,,	1920 Apr. 27	442·5	+1	383·7	32·5	32·6	−0·1
,,	May 4	449·5	,,	390·7	32·8	32·7	+0·1
5835	1921 Jan. 18	708·4	,,	649·6	38·8	37·0	+1·8
6243	1922 Mar. 15	23129·3	+2	4·5	40·4	38·2	+2·2
6247	,, 16	130·3	,,	5·5	38·3	38·2	+0·1
6251	,, 18	132·3	,,	7·5	37·2	38·1	−0·9
6254	,, 20	134·3	,,	9·5	38·2	38·1	+0·1
6258	,, 21	135·3	,,	10·5	39·3	38·0	+1·3
6262	,, 24	138·3	,,	13·5	36·0	37·9	−1·9
6746	1923 Dec. 5	23759·5	,,	634·7	34·7	36·7	−2·0
6779	1924 Jan. 8	793·4	,,	668·6	36·3	37·4	−1·1
6826	Feb. 22	838·3	,,	713·5	39·8	38·4	+1·4
6829	,, 27	843·3	,,	718·5	40·6	38·5	+2·1
7123	Dec. 4	24124·4	,,	999·6	39·6	39·8	−0·2
7126	,, 5	125·5	,,	1000·7	40·8	39·8	+1·0
7127	,, 6	126·5	,,	1001·7	+39·3	+39·7	−0·4

Preliminary elements were derived by Russell's method, adopting a period of 1070 days. The eccentricity was found to be small and a value of 0·08 was adopted. Normal equations for the correction of the preliminary elements were formed in the usual manner with the aid of Schlesinger's tables. It was found, however, that the coefficients of the terms depending on ω and T in these equations were so nearly proportional to one another that the corrections to these two elements were not separately determinable. The equations were therefore employed only to give an improved value of the period. The correction to the adopted period was found to be $-4{\cdot}0 \pm 4{\cdot}0$ days. The revised period of 1066·0 days was therefore adopted.

For small values of e, when considerable changes can be made in ω and the correlated value of T without altering much the resulting computed velocities, it is best to employ the analytical method, in which the velocity is expressed as a power series in the eccentricity. The most convenient expansion to use is the following :—

$$V = V_0 + K(1 - e^2)\cos(\omega + M) + Ke\cos(\omega + 2M),$$

in which M denotes the mean anomaly.

In this expansion all the terms in e^2 have not been included. The neglected terms in e^2 are about equal to half the total second order terms, as pointed out by Zurhellen,* and amount at most to Ke^2. In the present case, with K = 4 kms. and $e \not> 0{\cdot}1$, the adoption of the above approximate expansion is equivalent to the neglect of quantities of the maximum amount of 0·04 km./sec. in the velocities. As each velocity determination is subject to a probable error of the order of ± 1 km., the neglect of quantities of this order is justifiable.

It is more convenient to replace M by $M - M_0$, in which M now denotes the phase angle measured from an arbitrary initial epoch and M_0 denotes the phase angle at periastron (which remains to be determined) measured from the same epoch.

From the velocity curve given by the observations, preliminary values are adopted for V_0, K, and M_m, the phase angle of the point of maximum velocity, measured from the initial epoch. Then $\omega = -(M_m - M_0)$. The velocities are computed with these values and the velocity residuals, δV, derived. The equations of condition for solution by least squares may then be put into the form †

$$\delta V_0 + \cos M \,.\, x + \sin M \,.\, y + \cos 2M \,.\, z + \sin 2M \,.\, w = \delta V,$$

where

$$\begin{aligned} X + x &= K(1 - e^2) \cos(\omega - M_0) \\ Y + y &= -K(1 - e^2) \sin(\omega - M_0) \\ z &= Ke \cos(\omega - 2M_0) \\ w &= -Ke \sin(\omega - 2M_0) \end{aligned}$$

and $X = K \cos M_m$, $Y = K \sin M_m$ are computed from the adopted elements.

The values of x, y, z, w having been obtained by a least-squares solution, K, e, ω, and M_0 are derived by the above equations. The great advantage of this method is that the coefficients in the normal equation are not correlated with one another and each unknown is readily derived with its proper weight.

In the present instance, the velocity residuals were derived by adopting as preliminary values:

$$\begin{aligned} V_0 &= +36{\cdot}60 \text{ km./sec.} \\ K &= 4{\cdot}10 \text{ km./sec.} \\ M_m &= 186°{\cdot}0, \end{aligned}$$

the initial epoch being J.D. 2,420,259·0. The resulting normal equations were:

δV_0	x	y	z	w	
+ 55·00	− 5·34	+ 0·25	− 12·27	− 4·43	= − 16·00
	+ 21·37	− 2·21	+ 1·88	− 0·92	= + 0·47
		+ 33·63	− 1·17	− 7·22	= + 2·59
			+ 26·68	+ 2·83	= + 6·76
				+ 28·32	= + 10·08

* *A.N.*, **177**, 321, 1908.
† Paddock, *L.O.B.*, **8**, 160, 1915.

whence

$$\begin{aligned} \delta V_0 &= -0{\cdot}24 \pm 0{\cdot}14 \text{ km./sec.} \\ x &= -0{\cdot}016 \pm 0{\cdot}21 \\ y &= +0{\cdot}156 \pm 0{\cdot}165 \\ z &= +0{\cdot}115 \pm 0{\cdot}195 \\ w &= +0{\cdot}346 \pm 0{\cdot}186. \end{aligned}$$

From these values are derived the following values for K, e, ω, and M_0 :—

$$\begin{aligned} K &= 4{\cdot}14 \pm 0{\cdot}21 \text{ km./sec.} \\ e &= 0{\cdot}088 \pm 0{\cdot}045 \\ \omega &= 64^\circ{\cdot}0 \pm 35^\circ{\cdot}8 \\ M_0 &= 247^\circ{\cdot}8 \pm 31^\circ{\cdot}9. \end{aligned}$$

The phase angle, M_0, from the initial epoch corresponds to a time interval of 733·8 days, with a probable error of $\pm 94^d{\cdot}5$. Hence the time of periastron is J.D. 2,420,259·0 + 733·8.

The final elements are therefore :

$$\begin{aligned} P &= 1066{\cdot}0 \pm 4{\cdot}0 \text{ days} \\ V_0 &= +36{\cdot}36 \pm 0{\cdot}14 \text{ km./sec.} \\ K &= 4{\cdot}14 \pm 0{\cdot}21 \text{ km./sec.} \\ e &= 0{\cdot}088 \pm 0{\cdot}045 \\ \omega &= 64^\circ{\cdot}0 \pm 35^\circ{\cdot}8 \\ T &= \text{J.D. } 2{,}420{,}992{\cdot}8 \pm 94^d{\cdot}5. \end{aligned}$$

The probable errors are derived from the residuals from these elements, as given in the last column of Table II. The probable error corresponding to a single plate of unit weight is $\pm$ 0·95 km./sec.

The values of $a \sin i$ and of the mass function, f, which result from the above elements, are :

$$\begin{aligned} a \sin i &= 60{\cdot}5 \times 10^6 \text{ kms.} \\ f &= 0{\cdot}00776 \odot. \end{aligned}$$

The radial velocity curve is shown in fig. 2, together with the observed velocities.

λ *Hydræ.*

Boss 2706 α $10^h\ 5^m{\cdot}7$ $\delta - 11^\circ\ 52'$ (1900) $3^m{\cdot}83$ Ko.

The variable velocity of λ Hydræ was announced by Wright and Burns * in 1906, from observations secured at Lick in the years 1898 to 1906. The Lick observations comprise 16 plates obtained in the period 1898 to 1908. The Cape observations were commenced after the last of the Lick observations and comprise altogether 23 plates obtained in the years 1909 to 1925, but with a gap of nearly seven years between 1916 and 1923 in which no plates were obtained. The long interval of twenty-seven years between the first and last plates of the combined series is of value for fixing the period with relatively small uncertainty.

The observations are summarised in Table III., in which the arrangement of the columns is the same as in Table I. It was obvious on

* *L.O.B.*, **4**, 68, 1906.

plotting the observations, using for the period a provisional value of 1588 days, that the eccentricity was comparatively small, and in view of the small velocity range it was decided again to employ the analytical method of solution. A preliminary solution was made first, however,

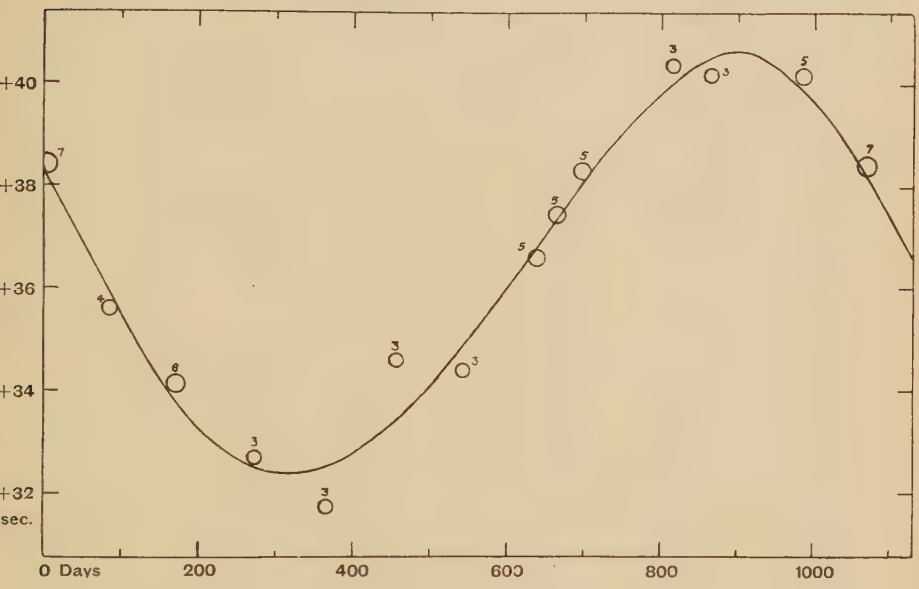


Fig. 2.—The Radial Velocity Curve of τ Puppis.

by the differential method in order to derive a more accurate value of the period. For this purpose the eccentricity was assumed to be 0·08, this value being derived by the Lehmann-Filhés method. The correction to the value of the period which was thus obtained was $-2{\cdot}2 \pm 5{\cdot}6$ days. The corrected period of 1585·8 days was definitely adopted.

Table III.

Observations of λ Hydræ.

Plate No.	Date.	J.D. 2,400,000+.	Revn. No.	Final Phase.	Observed Velocity.	Computed Velocity.	δV (O−C).
				d.	km./sec.	km./sec.	km./sec.
L	1898 Mar. 30	14379·7	−3	342·0	+23·0	+22·3	+0·7
,,	1899 Feb. 13	699·9	,,	662·2	22·6	22·0	+0·6
,,	1900 ,, 12	15063·9	,,	1026·2	18·3	18·3	0·0
,,	,, 26	077·8	,,	1040·1	19·2	18·2	+1·0
,,	Mar. 9	088·8	,,	1051·1	18·1	18·0	+0·1
,,	,, 12	091·7	,,	1054·0	19·0	18·0	+1·0
,,	,, 13	092·8	,,	1055·1	18·7	18·0	+0·7
,,	,, 27	106·7	,,	1069·0	+19·0	+17·8	+1·2

TABLE III.—*continued.*

Plate No.	Date.	J.D. 2,400,000+.	Revn. No.	Final Phase.	Observed Velocity.	Computed Velocity.	δV (O−C).
				d.	km./sec.	km./sec.	km./sec.
L	1900 Dec. 5	15359·1	−3	1321·4	+15·0	+15·4	−0·4
,,	1901 Jan. 15	401·0	,,	1363·3	17·0	15·3	+1·7
,,	Dec. 23	742·0	−2	118·5	21·0	19·3	+1·7
,,	1902 Dec. 31	16115·0	,,	491·5	24·0	22·7	+1·3
,,	1903 Feb. 23	169·9	,,	546·4	22·4	22·6	−0·2
,,	1904 Mar. 31	571·8	,,	948·3	20·2	19·4	+0·8
,,	1906 May 3	17334·7	−1	125·4	20·0	19·4	+0·6
,,	1908 Apr. 19	18051·7	,,	842·4	21·0	20·4	+0·6
2143	1909 Jan. 20	327·6	,,	1118·3	17·4	17·2	+0·2
2148	,, 21	328·6	,,	1119·3	15·5	17·2	−1·7
2155	,, 22	329·6	,,	1120·3	17·7	17·2	+0·5
2902	1910 Dec. 2	19008·6	0	213·5	20·9	20·9	0·0
3036	1911 Mar. 2	098·4	,,	303·3	20·5	22·0	−1·5
3531	1912 Mar. 11	473·4	,,	678·3	21·8	21·9	−0·1
3926	Dec. 18	795·6	,,	1000·5	15·8	18·6	−2·8
4766	1916 May 10	20994·3	+1	613·4	19·7	22·3	−2·6
4770	,, 15	999·3	,,	618·4	20·0	22·3	−2·3
6500	1923 Feb. 1	23452·5	+2	1485·8	17·3	15·7	+1·6
6514	,, 7	458·5	,,	1491·8	18·3	15·8	+2·5
6522	,, 12	463·4	,,	1496·7	17·4	15·8	+1·6
6525	,, 13	464·4	,,	1497·7	15·7	15·9	−0·2
6556	Mar. 5	484·4	,,	1517·7	14·1	16·1	−2·0
6559	,, 7	486·4	,,	1519·7	14·5	16·1	−1·6
6583	Apr. 13	523·4	,,	1556·7	14·3	16·6	−2·3
6831	1924 Feb. 27	843·4	+3	290·9	21·6	21·9	−0·3
6833	,, 28	844·4	,,	291·9	23·5	21·9	+1·6
6874	Mar. 19	864·4	,,	311·9	20·1	22·1	−2·0
6900	Apr. 4	880·4	,,	327·9	22·9	22·2	+0·7
7147	1925 Jan. 6	24157·7	,,	605·2	24·5	22·4	+2·1
7167	,, 22	173·5	,,	621·0	23·9	22·3	+1·6
7169	,, 26	177·5	,,	625·0	+23·3	+22·3	+1·0

Using this period, the observations were collected into twelve normal places, details of which are contained in Table IV. The first and second columns give the number of plates on which each normal place is based and the adopted weight respectively. Unit weight is assigned to the mean of 5 plates. The third column contains the mean observed velocity. The fourth column gives the phase angles reckoned from the initial epoch given below. The residuals used for the formation of the constant terms of the normal equations are given in the fifth column.

The remaining three columns contain the phases, velocities, and residuals which result from the final elements.

TABLE IV.

λ Hydræ : Normal Places.

No. of Plates.	Weight.	Observed Velocity.	Phase Angle.	δV (O−C) Preliminary.	Final Phase.	Computed Velocity.	δV (O−C) Final.
		km./sec.		km./sec.	d.	km./sec.	km./sec.
6	1·20	+21·93	11·67	+0·08	311·3	+22·20	−0·27
2	0·40	23·20	58·80	+0·01	518·9	22·82	+0·38
5	1·00	22·28	80·98	−0·74	616·6	22·44	−0·16
2	0·40	22·20	93·14	−0·51	670·2	22·10	+0·10
1	0·20	21·00	132·24	+0·13	842·4	20·58	+0·42
2	0·40	18·00	162·20	−1·02	974·4	19·07	−1·07
2	0·40	18·75	175·53	+0·53	1033·2	18·36	+0·39
4	0·80	18·70	181·02	+0·79	1057·3	18·08	+0·62
3	0·60	16·87	195·10	−0·32	1119·3	17·34	−0·43
2	0·40	16·00	245·74	0·00	1342·2	15·46	+0·54
7	1·40	15·94	283·66	−0·93	1509·4	16·11	−0·17
3	0·60	+20·63	335·62	+0·86	152·5	+20·03	+0·60

For the application of Zurhellen's method the adopted quantities were as follows :—

$$V_0 = +19{\cdot}60 \text{ km./sec.}$$
$$K = 3{\cdot}60 \text{ km./sec.}$$
$$M_m = 63°{\cdot}0,$$

with initial epoch J.D. 2,419,055·0. The resulting normal equations were :

δV_0	x	y	z	w	
+ 7·80	− 0·06	+ 0·13	+ 0·47	+ 0·14	= − 1·35
	+ 4·14	+ 0·07	− 0·96	+ 0·11	= − 0·14
		+ 3·66	− 0·02	+ 0·89	= + 0·10
			+ 5·84	+ 0·15	= + 2·83
				+ 1·96	= + 0·14,

whence

$$\delta V_0 = -0{\cdot}205 \pm 0{\cdot}11$$
$$x = +0{\cdot}081 \pm 0{\cdot}16$$
$$y = +0{\cdot}029 \pm 0{\cdot}17$$
$$z = +0{\cdot}514 \pm 0{\cdot}13$$
$$w = +0{\cdot}028 \pm 0{\cdot}24.$$

From these values are derived the following values for K, e, ω, and M_0 :—

$$K = 3{\cdot}74 \pm 0{\cdot}17 \text{ km./sec.}$$
$$e = 0{\cdot}138 \pm 0{\cdot}037$$
$$\omega = 238°{\cdot}9 \pm 27°{\cdot}0$$
$$M_0 = 59°{\cdot}0 \pm 26°{\cdot}6.$$

The phase angle, M_0, from the adopted initial epoch corresponds to a time interval of − 259·9 days, with a probable error of ± 117^d·3. Hence the time of periastron is J.D. 2,418,795·1.

The probable errors are based upon the final residuals given in Table IV. The probable error corresponding to unit weight (mean of 5 plates) is ± 0·315 km./sec.

It may be mentioned that Zurhellen's method is usually restricted to orbits with an eccentricity less than 0·10, owing to the neglect of terms of an order not exceeding Ke^2. Its use is legitimate in the present instance on account of the small amplitude of the velocity variation.

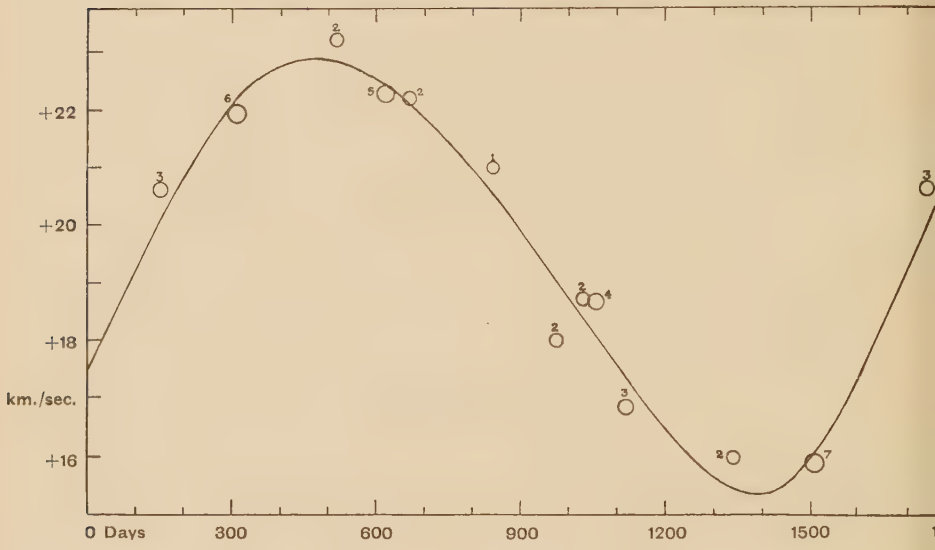


FIG. 3.—The Radial Velocity Curve of λ Hydræ.

The value of Ke^2 is 0·07, so that the terms which have been neglected amount only to a few hundredths of a kilometre.

The final elements are :

$$\begin{aligned} P &= 1585{\cdot}8 \pm 5{\cdot}6 \text{ days} \\ V_0 &= +19{\cdot}39 \pm 0{\cdot}11 \text{ km./sec.} \\ K &= 3{\cdot}74 \pm 0{\cdot}17 \text{ km./sec.} \\ e &= 0{\cdot}138 \pm 0{\cdot}037 \\ \omega &= 238^\circ{\cdot}9 \pm 27^\circ{\cdot}0 \\ T &= \text{J.D. } 2{,}418{,}795{\cdot}1 \pm 117^d{\cdot}3. \end{aligned}$$

The values of $a \sin i$ and of the mass function, f, which result from the above elements, are :

$$\begin{aligned} a \sin i &= 80{\cdot}8 \times 10^6 \text{ kms.} \\ f &= 0{\cdot}00837 \odot. \end{aligned}$$

The radial velocity curve is shown in fig. 3.

α Tucanæ.

Boss 5747 α 22^h 11^m·7 δ − 60° 45′ 2^m·91 K2.

The variable radial velocity of α Tucanæ was announced by Paddock * from observations secured at the D.O. Mills Station of the Lick Observatory between 1903 and 1910. The star was already known to have variable velocity at the Cape Observatory, two plates secured in 1908 and 1910 having shown a velocity difference of over 7 km./sec. The Lick observations comprise 17 plates secured between the years 1903 and 1920. The Cape observations comprise 14 plates secured between the years 1908 and 1924. The combined series, extending from 1903 to 1924, does not quite cover two complete periods. The period is therefore necessarily somewhat uncertain.

The observations are summarised in Table V. The following preliminary elements were derived by the use of Russell's method :—

$$\begin{aligned} P &= 4300 \text{ days} \\ G &= +43{\cdot}60 \text{ km./sec.} \\ K &= 7{\cdot}20 \text{ km./sec.} \\ e &= 0{\cdot}34 \\ \omega &= 43^\circ \\ T &= \text{J.D. } 2{,}418{,}610{\cdot}0. \end{aligned}$$

TABLE V.

Observations of α Tucanæ.

Plate No.	Date.	J.D. 2,400,000+.	Revn. No.	Final Phase.	Observed Velocity.	Computed Velocity.	δV (O−C).
				d.	km./sec.	km./sec.	km./sec.
L	1903 Sept. 20	16378·6	−1	1909·9	+38·3	+38·6	−0·3
,,	1904 ,, 10	734·7	,,	2266·0	40·0	40·0	0·0
,,	Oct. 18	772·6	,,	2303·9	40·1	40·1	0·0
,,	1905 July 16	17043·8	,,	2575·1	39·9	41·4	−1·5
,,	,, 22	049·9	,,	2581·2	40·0	41·5	−1·5
,,	Sept. 2	091·7	,,	2623·0	41·4	41·7	−0·3
,,	1907 ,, 7	826·7	,,	3358·0	46·9	46·8	+0·1
1960	1908 Oct. 16	18231·4	,,	3762·7	49·7	50·5	−0·8
2823	1910 Sept. 20	935·3	0	268·9	42·4	42·8	−0·4
L	,, 30	945·6	,,	279·2	44·4	42·6	+1·8
2847	Oct. 21	966·4	,,	300·0	41·6	42·1	−0·5
L	,, 30	975·6	,,	309·2	41·6	42·0	−0·4
3224	1911 July 16	19234·7	,,	568·3	36·5	38·5	−2·0
3232	,, 26	244·6	,,	578·2	36·7	38·4	−1·7
3292	Oct. 3	313·4	,,	647·0	37·4	37·9	−0·5
3343	Nov. 7	348·4	,,	682·0	36·2	37·7	−1·5
3702	1912 July 9	593·7	,,	927·3	38·2	36·9	+1·3
L	1914 ,, 29	20343·8	,,	1677·4	+37·7	+37·9	−0·2

* *L.O.B.*, **6**, 152, 1911.

TABLE V.—*continued.*

Plate No.	Date.	J.D. 2,400,000+.	Revn. No.	Final Phase.	Observed Velocity.	Computed Velocity.	δV (O−C).
				d.	km./sec.	km./sec.	km./sec.
L	1918 Aug. 16	21822·8	0	3156·4	+46·9	+45·2	+1·7
,,	Nov. 23	921·6	,,	3255·2	46·7	45·9	+0·8
,,	,, 28	926·5	,,	3260·1	46·2	46·0	+0·2
,,	1920 May 11	22456·9	,,	3790·5	50·3	50·7	−0·4
,,	June 2	478·9	,,	3812·5	51·2	50·8	+0·4
,,	July 22	528·8	,,	3862·4	51·5	51·1	+0·4
,,	,, 25	531·8	,,	3865·4	50·8	51·1	−0·3
6408	1922 Oct. 9	23337·4	+1	473·3	40·8	39·5	+1·3
6721	1923 ,, 17	710·4	,,	846·3	35·7	37·1	−1·4
6726	,, 22	715·4	,,	851·3	38·2	37·1	+1·1
6729	,, 30	723·4	,,	859·3	37·0	37·1	−0·1
6735	Nov. 8	732·4	,,	868·3	37·1	37·0	+0·1
7089	1924 Oct. 21	24080·5	,,	1216·4	+39·3	+36·9	+2·4

Normal equations, based on these elements, were formed with the aid of Schlesinger's tables, for the derivation of differential corrections. The normal equations were :

δG	δK	π	ϵ	γ	m	
+ 31·00	− 7·68	− 1·40	+ 13·91	+ 1·82	+ 1·10	= + 4·35
	+ 14·40	− 5·03	− 2·41	− 3·68	− 2·90	= + 3·62
		+ 16·60	+ 2·53	+ 8·21	+ 2·46	= − 8·62
			+ 8·32	+ 2·48	+ 0·66	= + 0·22
				+ 4·76	+ 1·52	= − 4·06
					+ 1·57	= − 3·20

The unit of time used in the formation of the coefficients of m was 4000 days.

The solution of the normal equations gave :

$$\delta G = + 0{\cdot}48 \pm 0{\cdot}40$$
$$\delta K = + 0{\cdot}03 \pm 0{\cdot}34$$
$$\pi = - 0{\cdot}685 \pm 0{\cdot}63 \qquad \delta\omega = + 5{\cdot}5^\circ \pm 5{\cdot}0^\circ$$
$$\epsilon = - 0{\cdot}813 \pm 0{\cdot}81 \qquad \delta e = + 0{\cdot}045 \pm 0{\cdot}045$$
$$\tau = + 1{\cdot}28 \pm 1{\cdot}39 \qquad \delta T = + 56{\cdot}4 \pm 61{\cdot}2$$
$$m = - 2{\cdot}16 \pm 0{\cdot}88 \qquad \delta P = - 102{\cdot}3 \pm 41{\cdot}6.$$

The final elements are therefore :

$$P = 4197{\cdot}7 \pm 41{\cdot}6 \text{ days}$$
$$G = + 44{\cdot}08 \pm 0{\cdot}40 \text{ km./sec.}$$
$$K = 7{\cdot}23 \pm 0{\cdot}34 \text{ km./sec.}$$
$$e = 0{\cdot}385 \pm 0{\cdot}045$$
$$\omega = 48^\circ{\cdot}5 \pm 5^\circ{\cdot}0$$
$$T = \text{J.D. } 2{,}418{,}666{\cdot}4 \pm 61^d{\cdot}2.$$

The probable errors are based upon the probable error for a single plate of ± 0·795 km./sec., derived from the final residuals.

The values for V_0, $a \sin i$, and f, which result from the above elements, are as follows :—

$$V_0 = +42{\cdot}24 \text{ km./sec.}$$
$$a \sin i = 385{\cdot}1 \times 10^6 \text{ kms.}$$
$$f = 0{\cdot}1295\ \odot.$$

The only spectroscopic binaries with a larger value of $a \sin i$ are ν Geminorum, ϵ Hydræ (visual binary), and ξ Pavonis.

The radial velocity curve is shown in fig. 4.

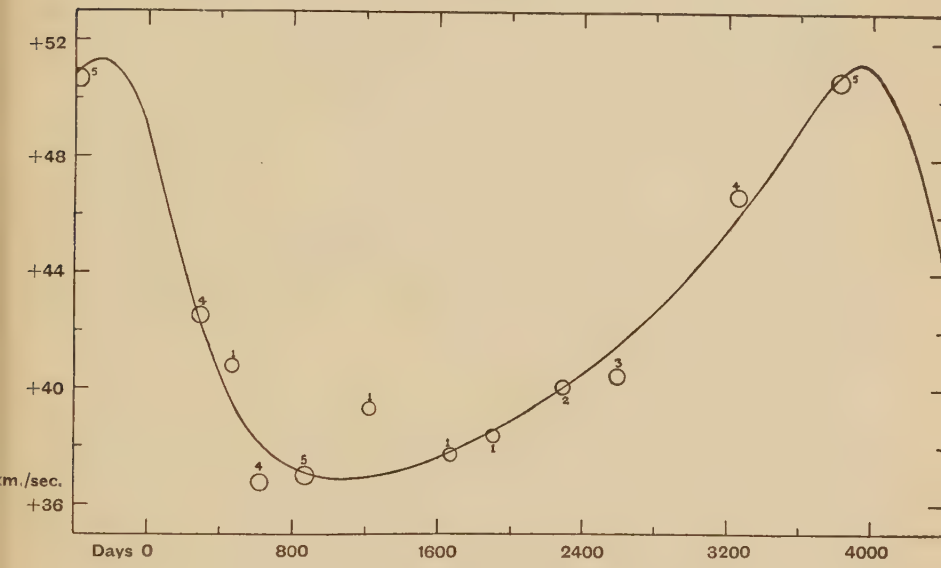


FIG. 4.—The Radial Velocity Curve of α Tucanæ.

Summary.

The orbits of the spectroscopic binaries β Reticuli, τ Puppis, λ Hydræ, and α Tucanæ have been derived by combining the observations secured at the Lick and Cape observatories. The four stars are of spectral type K and their velocity variations are characterised by small amplitude and long period. The principal elements are as follows :—

	β Ret.	τ Pup.	λ Hyd.	α Tuc.
P (days)	1911·5	1066·0	1585·8	4197·7
V_0 (km./sec.)	+51·11	+36·36	+19·39	+42·24
K (km./sec.)	5·19	4·14	3·74	7·23
e	0·210	0·088	0·138	0·385
ω	13°·8	64°·0	238°·9	48°·5
$a \sin i$ (10^6 km.)	133·4	60·5	80·8	385·1
$f(\odot)$	0·0260	0·0078	0·0084	0·1295

Royal Observatory, Cape of Good Hope :
1928 *May* 4.

The Radial Velocity Variations of α Orionis and α Scorpii. By H. Spencer Jones, M.A., Sc.D., H.M. Astronomer.

The two giant red stars, α Orionis and α Scorpii, have variable radial velocities, the total range of variation being of the order of 4 kms. per sec. in each case. The representation of the velocity variations by orbital elements (which does not necessarily imply that these stars are binary systems) is difficult on account of the small velocity range. The orbital elements which are given for these two stars in the " Third Catalogue of Spectroscopic Binary Stars " * were published in 1911 and 1909 respectively, and it has been shown by Lunt † that in neither case do the orbital elements satisfy later observations.

Orbital elements for α Orionis were derived by Bottlinger.‡ The material available for the derivation of the period of velocity variation was not sufficient to determine it with any accuracy ; the maximum and minimum points of the velocity curve were very weakly determined. The only orbital elements for α Scorpii which have been published were derived by Halm § in order to obtain a representation of the velocity variations during the period covered by the observations used for the determination of the constant of aberration. A large portion of the velocity curve depended upon a small number of earlier Lick plates, and subsequent observations showed that the adopted period was seriously in error.

The recent Lick Publication,‖ " Radial Velocities of Stars brighter than Visual Magnitude 5·51," contains, in addition to radial velocity determinations of α Orionis and α Scorpii which had been published previously in *Lick Bulletins*, a large number of hitherto unpublished determinations. In this volume are given the measures of 89 spectrograms of α Orionis secured between the years 1896 and 1926, and of 48 spectrograms of α Scorpii secured between the years 1897 and 1922.

In view of the importance for theoretical purposes of a satisfactory determination of the radial velocity variations of these two giant stars, and in view of the variations of angular diameter of α Orionis observed at Mount Wilson with the interferometer, it seemed desirable to combine the Lick results with the extensive Cape observations and to derive new orbital elements for the two stars. Results obtained at other observatories have not been incorporated ; Bottlinger utilised observations obtained both at Bonn and at Potsdam for deriving the orbital elements of α Orionis. Systematic differences between the results obtained at different observatories which are of relatively small importance when dealing with a large velocity variation become of great importance when the velocity variation is as small as in the case of the two stars under consideration. Such systematic differences cannot be determined with accuracy from a small number of observations, and it therefore seemed preferable to leave out of consideration the observa-

* *L.O.B.*, **11**, No. 355, 1924.
‡ *A.N.*, **187**, 33, 1911.
‖ *L.O.P.*, **16**, 1928.
† *Ap. J.*, **44**, 250, 1916.
§ *Cape Annals*, **10**, pt. 3, 56*c*, 1909.

tions secured at Bonn, Potsdam, and elsewhere, and to base the determination entirely upon the two main series of observations.

The observations published in the Lick volume were obtained partly at Mount Hamilton and partly at Santiago, and the spectrographs and dispersions used at either station have not been uniform throughout. In preparing the results for publication, however, all the results have as far as possible been reduced to a homogeneous system (*vide* Introduction, pp. xxi.–xxx.). The Cape observations were all obtained with the long camera attached to the four-prism spectrograph. Some of the Cape plates were measured by two or three different methods; only the measures made with the Hartmann spectrocomparator have been utilised as the accidental errors of these measurements, which are differential in nature, are considerably smaller than those of the measurements by the other methods which were used.

α Orionis.

The observations, as published in the *Cape Annals*, **10**, and the *Lick Observatory Publications*, **16**, are summarised in Table I., mean velocities and epochs being given for observations secured near the same epoch.

TABLE I.

Radial Velocity Observations of α Orionis.

Cape.			Lick.		
Mean Date.	Mean Velocity. km./sec.	No. of Obs.	Mean Date.	Mean Velocity. km./sec.	No. of Obs.
1903·88	+18·60	1	1896·90	+18·21	3
04·12	19·40	1	98·10	17·76	2
06·12	24·00	2	1903·85	18·89	4
06·19	22·56	10	06·07	22·81	1
06·75	22·11	15	07·94	20·49	20
07·22	21·27	12	08·27	20·32	9
08·07	20·27	22	10·00	18·65	5
08·22	20·28	16	11·15	21·38	3
08·72	20·62	4	11·74	21·31	1
09·11	18·01	11	12·02	26·29	3
09·77	19·27	4	13·73	20·16	1
10·16	19·98	6	17·14	23·29	3
10·73	20·27	6	18·08	21·21	2
11·16	24·17	9	19·02	20·61	1
11·70	22·14	5	20·99	19·17	5
12·17	23·47	7	21·78	21·77	6
12·75	20·90	8	22·02	19·61	5
13·11	+24·40	2	23·21	24·31	1
			26·04	20·51	8

The Cape observation of 1912 February 23 and the Lick observation of 1909 November 26 have been rejected, both being obviously considerably in error. Measures of 4 spectrograms obtained at Santiago in 1922 with one-prism dispersion have not been utilised. The remaining Mount Hamilton and Santiago spectrograms were secured with two- or three-prism dispersion. In all, the measures of 141 spectrograms obtained at the Cape and of 83 spectrograms obtained at Lick have been utilised. In order to bring the Lick measures into the best general agreement with the Cape measures, a correction of $-0{\cdot}39$ km./sec. has been applied to the published Lick velocities. This correction is incorporated in the mean velocities given in Table I.

These observations were plotted graphically and a period of 5·79 years was adopted from the graph as a first approximation to the period. Bottlinger's period was 6·0 years. With this period all the observations were reduced to one revolution. The correctness of this procedure may be queried as, in view of the irregular range and period of the light variations of Betelgeuse, it might be anticipated that the radial velocity curve would also show irregularities of a corresponding nature. Whether this is so or not, the preliminary plot of the observations showed no definite evidence of such irregularities. It is therefore desirable to ascertain to what extent the complete series of observations can be represented by one set of elements.

Having reduced all the observations to one revolution, using the period of 5·79 years, a curve was drawn so as to represent them as well as possible, and from this curve preliminary elements were derived by the method of Lehmann-Filhés.*

$$\begin{aligned} K &= 2{\cdot}13 \text{ km./sec.} \\ V_0 &= +21{\cdot}04 \text{ km./sec.} \\ e &= 0{\cdot}25 \\ \omega &= 270^\circ \\ T &= 1904{\cdot}73 \\ P &= 5{\cdot}79 \text{ years.} \end{aligned}$$

Lick and Cape observations of about the same epoch were combined to give 29 normal places, arranged in order of date. Details of the normal places are given in Table II. These were used for a least-squares solution to derive corrections to the adopted elements. In cases such as this, with a small velocity range, the elements can be varied appreciably without greatly altering the sum of the squares of the residuals. A least-squares adjustment is necessary in order to derive the elements which give the best representation of the observations. By this means also probable errors can be assigned to each element.

* *A.N.*, **136**, 17, 1894.

TABLE II.

Observations of α Orionis : Normal Places.

Date.	Mean Observed Velocity.	No. of Plates.	Revolution.	Final Phase.	Computed Velocity.	δV (O−C).	Weight.
	km./sec.			*y.*	km./sec.	km./sec.	
1896·90	+18·21	3	−2	3·49	+19·81	−1·60	0·6
98·10	17·76	2	,,	4·69	19·17	−1·41	0·4
1903·86	18·91	5	−1	4·67	19·16	−0·25	1·0
04·12	19·40	1	,,	4·93	19·40	0·00	0·2
06·17	22·80	13	0	1·20	23·04	−0·24	2·6
06·75	22·11	15	,,	1·78	22·31	−0·20	3·0
07·22	21·27	12	,,	2·25	21·58	−0·31	2·4
07·94	20·49	20	,,	2·97	20·50	−0·01	4·0
08·07	20·27	22	,,	3·10	20·31	−0·04	4·4
08·24	20·29	25	,,	3·27	20·08	+0·21	5·0
08·72	20·62	4	,,	3·75	19·52	+1·10	0·8
09·11	18·01	11	,,	4·14	19·21	−1·20	2·2
09·77	19·27	4	,,	4·80	19·25	+0·02	0·8
10·09	19·38	11	,,	5·12	19·73	−0·35	2·2
10·73	20·27	6	,,	5·76	21·66	−1·39	1·2
11·16	23·47	12	+1	0·41	22·85	+0·62	2·4
11·71	22·00	6	,,	0·96	23·21	−1·21	1·2
12·12	24·32	10	,,	1·37	22·86	+1·46	2·0
12·75	20·90	8	,,	2·00	21·97	−1·07	1·6
13·11	24·40	2	,,	2·36	21·41	+2·99	0·4
13·73	20·55	1	,,	2·98	20·74	−0·19	0·2
17·14	23·68	3	+2	0·61	23·13	+0·55	0·6
18·08	21·60	2	,,	1·55	22·63	−1·03	0·4
19·02	21·00	1	,,	2·49	21·21	−0·21	0·2
20·99	19·56	5	,,	4·46	19·11	+0·45	1·0
21·78	22·16	6	,,	5·25	20·03	+2·13	1·2
22·02	20·00	5	,,	5·49	20·74	−0·74	1·0
23·21	24·70	1	+3	0·90	23·22	+1·48	0·2
26·04	20·90	8	,,	3·73	29·55	+1·35	1·6

The velocity residuals resulting from the above elements were computed, and normal equations were formed with the aid of Schlesinger's Tables.* Using Schlesinger's notation for the definition of the quantities π, ϵ, τ, and m,† which determine the corrections $\delta\varpi$, δe, δT, and δP respectively, the normal equations are :—

δG	δK	π	ε	τ	m	
+ 44·800	− 1·897	+ 19·456	− 0·065	+ 4·588	− 3·178	= − 0·73
	+ 18·298	− 0·350	+ 2·826	+ 0·451	+ 0·111	= − 0·23
		+ 26·502	− 1·920	+ 12·775	+ 0·830	= − 1·09
			+ 4·439	− 1·062	− 0·628	= + 0·89
				+ 7·290	+ 1·487	= − 0·21
					+ 3·010	= + 0·19.

* *Allegheny Publications*, **1**, No. 6.

† *Loc. cit.*, p. 35.

In these equations the unit of time used for the formation of the coefficients of m is 10 years.

The solution of these equations gives :—

$$\begin{aligned} \delta G &= +0{\cdot}133 \pm 0{\cdot}13 \\ \delta K &= -0{\cdot}066 \pm 0{\cdot}09 \\ \pi &= -0{\cdot}586 \pm 0{\cdot}41 \\ \epsilon &= +0{\cdot}212 \pm 0{\cdot}20 \\ \tau &= +0{\cdot}960 \pm 0{\cdot}72 \\ m &= -0{\cdot}064 \pm 0{\cdot}28 \end{aligned} \qquad \begin{aligned} \delta\varpi &= +15{\cdot}8^\circ \pm 11{\cdot}1^\circ \\ \delta e &= -0{\cdot}042 \pm 0{\cdot}039 \\ \delta T &= +0{\cdot}241 \pm 0{\cdot}18 \\ \delta P &= -0{\cdot}009 \pm 0{\cdot}040. \end{aligned}$$

These corrections would probably be modified slightly by carrying out a second least squares solution as a further approximation. In view, however, of the fact that the correction to each element is of the same order as the probable error attaching to the element, little improvement would result from a second approximation.

The following were therefore adopted as the elements which provide the best representation of the Cape and Lick observations, Bottlinger's elements being given alongside for comparison :—

	New Elements.	Bottlinger's Elements.
G	$+ 21{\cdot}17 \pm 0{\cdot}13$ km./sec.	
K	$2{\cdot}06 \pm 0{\cdot}09$ km./sec.	2·45 km./sec.
e	$0{\cdot}208 \pm 0{\cdot}039$	0·24
ω	$285^\circ{\cdot}8 \pm 11^\circ{\cdot}1$	255°
T	$1904{\cdot}97 \pm 0^y{\cdot}18$	1904·58
P	$5{\cdot}781 \pm 0{\cdot}040$ years	6·0 years
V_0	$+ 21{\cdot}05 \pm 0{\cdot}15$ km./sec.	+ 21·3 km./sec.

It will be seen that Bottlinger's elements are in general agreement with those here derived, though the amplitude of the velocity variation has been appreciably reduced.

The probable errors given above have been derived from the squares of the velocity residuals for the single plates by the usual formula. Equal weight has been given to each Lick and Cape plate, and the probable error for a single plate is found to be $\pm 0{\cdot}84$ km./sec. The representation of the observations by the above elements is shown in fig. 1 in which the observations have been grouped according to phase, irrespective of the revolution number.

The values of the projected semi-axis major and of the mass-function which result from the above elements are :—

$$a \sin i = (58{\cdot}5 \pm 2{\cdot}6) \times 10^6 \text{ kms.}$$
$$f = 0{\cdot}00179\,\odot.$$

The residuals for the normal places are given in the second last column of Table II. An inspection of these residuals does not reveal any marked run. The earlier residuals are predominantly negative, but are small in amount, and in view of the magnitude of most of the residuals, the run in the sign of the residuals is probably without significance. For the period under consideration there is no clear indication of any change

in period or of amplitude of variation. Observations extending over a considerably longer period would be required in order to justify this assertion as a general statement, and it is to be hoped that this star will be retained under investigation at observatories which are engaged upon radial velocity observations. It is not improbable that the period may be irregular, though maintaining a steady average value as in the case of the sun-spot period. A large majority of the observations which are available were secured in the interval 1906 to 1912, and the observations made before and after this period are not sufficiently numerous to settle the question whether the period is variable.

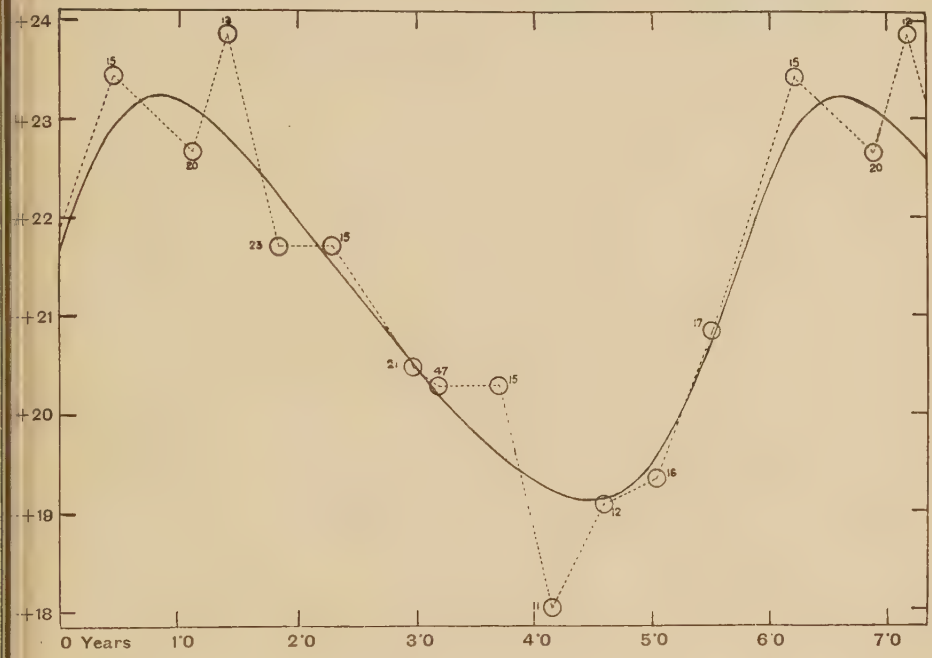


Fig. 1.—The Radial Velocity Curve of α Orionis.

The residuals in Table II. lead to a probable error for unit weight (mean of 5 plates) of ± 0·80 km./sec., corresponding to a probable error for a single plate of ± 1·8 km./sec. This quantity is unexpectedly large. α Orionis has a spectrum with good lines well adapted for measurement and, being a bright star, there is no danger of underexposed spectra. The Cape plates were all obtained with the four-prism spectrograph in conjunction with the long camera and have a scale of 8·4 Angstrom units per mm. With plates of good spectra having so great a dispersion measured differentially in the Hartmann spectrocomparator, the probable error of a single plate should not exceed ± 0·5 km./sec., this figure including both plate errors and errors of measurement. For the Lick plates, taken with two- or three-prism

dispersion, the probable error of a single plate should not exceed ± 0·75 km./sec. We have seen that the probable error of a single plate which results from the residuals of the individual plate is ± 0·84 km./sec. Although this value is larger than would be anticipated, it is small compared with the very large probable error of ± 1·8 km./sec., derived from the residuals of the normal places. Such a probable error would be anticipated only for a spectrum with poor lines and low dispersion.

This high value is undoubtedly due to short-period fluctuations in velocity. An examination of some of the larger residuals shows this clearly. A few examples may be given in illustration. Details of these are given in Table III.; two groups of Cape observations and two of Lick observations have been selected, the mean of each group having been used to give one normal place in the least-squares solution. The

TABLE III.

Cape.

Date.	V. km./sec.
1909 Jan. 13	+17·6
14	16·9
21	17·3
23	16·8
Feb. 6	18·5
15	18·2
16	19·4
25	17·4
Mar. 2	19·1
10	18·2
19	18·7
P.E. single plate	± 0·59
P.E. of mean	± 0·18
Resid. from curve	− 1·20

Lick.

Date.	V. km./sec.
1921 Sept. 28	+21·6
Oct. 4	23·1
4	23·0
17	22·1
17	22·3
26	20·7
P.E. single plate	± 0·61
P.E. of mean	± 0·25
Resid. from curve	+ 2·13

Cape.

Date.	V. km./sec.
1912 Sept. 6	+20·7
10	20·0
29	21·6
30	22·6
Oct. 4	21·3
14	20·6
15	20·4
23	20·0
P.E. single plate	± 0·60
P.E. of mean	± 0·21
Resid. from curve	− 1·07

Lick.

Date.	V. km./sec.
1925 Nov. 13	+17·7
18	18·1
Dec. 30	23·0
1926 Jan. 14	21·7
21	22·5
21	21·5
Mar. 5	21·3
5	21·4
P.E. single plate	±1·31 ±0·47 *
P.E. of mean	±0·46 ±0·19 *
Resid. from curve	+1·35

(* Last 6 plates only.)

dates of each plate in these four groups and the measured velocities are given. Below each group is given the probable error of a single plate derived from the internal agreement of the group, calculated from the squares of the residuals from the mean velocity; the probable error of the mean velocity and the residual form the mean curve. It will be seen that the residual is in each case several times greater than the probable error of the mean value. The last example is of particular interest; the observations used for this normal place cover a longer period than was the case for any other normal place. The first two observations are separated from the next by an interval of six weeks, and during this period a real velocity change of about 4 km./sec. seems to have occurred. The probable errors derived for the last six plates only are given separately and are in better accordance with the other values than those which are obtained from the whole group.

An estimate of the magnitude of the short-period fluctuations may be made. The Cape and Lick plates will be considered separately. From the squares of the residuals of individual plates the total probable error of a single Cape plate is found to be $\pm$ 0·73, and of a single Lick plate $\pm$ 0·99. These probable errors are compounded of the errors arising from the short-period fluctuations and the normal errors of the plate and of measurement. If differences of residuals of consecutive plates taken at one epoch are formed, the consecutive plates being secured at an interval of a few days at most, the short-period fluctuation will be to a large extent eliminated, as we have no evidence in support of velocity changes with a period of a few days only. In this way the probable error of a difference between two consecutive Cape plates is found to be $\pm$ 0·74, and between two consecutive Lick plates is found to be $\pm$ 0·78. If then we denote by σ_C, σ_L the probable errors (plate errors and measurement errors) of a single Cape and Lick plate respectively, and by σ the probable error arising from the short-period variations, we have

$$\begin{aligned} \sigma_C^2 + \sigma^2 &= 0{\cdot}53 \\ \sigma_L^2 + \sigma^2 &= 0{\cdot}98 \\ 2\sigma_C^2 &= 0{\cdot}55 \\ 2\sigma_L^2 &= 0{\cdot}61 \end{aligned}$$

whence

$$\sigma_C^2 = 0{\cdot}23, \qquad \sigma_L^2 = 0{\cdot}37, \qquad \sigma^2 = 0{\cdot}46.$$

The equations are not entirely consistent, but this was hardly to be expected as the Lick plates are distributed differently in time from the Cape plates, and the effect of the short-period fluctuations will not be the same for the two series. We thus conclude that :—

Probable error of one Cape plate	= $\pm$ 0·48	km./sec.
,, ,, ,, Lick ,,	= $\pm$ 0·61	,,
,, ,, due to fluctuations	= $\pm$ 0·68	,,

The probable errors for the Cape and Lick plates derived in this way are in accord with what would be anticipated from the dispersions employed. It will be noticed that the probable error due to the fluctua-

tions is greater than that due to combined plate-errors and errors of measurement. In utilising residuals from the normal places to deduce probable errors, the probable error of a normal place based upon the mean of n plates, secured within a short interval of time and for which the fluctuation may be assumed constant, is

$$\left(\sigma^2 + \frac{\sigma_0^2}{n}\right)^{\frac{1}{2}},$$

and however great n may be, the probable error will never be less than σ. Each normal place is therefore liable to a probable error which is greater than σ, whatever its weight may be. For deriving probable errors of the orbital elements of the long-period variation, it is the residuals from the single plates and not those from the normal places which should be used.

The fluctuations are apparently irregular in period and amplitude; the period seems to be of the order of a few months. This may be illustrated by a part of the Cape series :—

1910·63 to 1910·82	6 plates.	All residuals −
1911·13 ,, 1911·19	9 ,,	,, ,, +
1911·66 ,, 1911·75	5 ,,	,, ,, −
1912·10 ,, 1912·24	8 ,,	7 residuals +, 1 residual −
1912·68 ,, 1912·81	8 ,,	7 ,, −, 1 ,, +
1913·09 ,, 1913·12	2 ,,	2 ,, +.

α Orionis is known to be an irregular variable star ; on account of its brightness and red colour, magnitude estimates are liable to be uncertain. Sir John Herschel concluded that the period of light variation was 6 months : Argelander deduced a period of 196 days. The long series of observations by Baxandall, which were reduced by Plummer,* give evidence of fluctuations in brightness of a few months superposed on a variation of longer period. Plummer was unable to detect in these observations the period of 6 years which Bottlinger derived for the radial velocity variations and suspected to exist in the light variations also. Some observations by Stebbins † with the selenium photometer definitely show short-period variations, with an estimated period of about 250 days, superposed on a variation of longer period, approximately 6 years.

The question arises whether a correlation exists between the long-period variations in brightness and in radial velocity, as suggested by Bottlinger. For deciding this question, observations both of radial velocity and of magnitude for the same time-interval are necessary. The only extended series of magnitude determinations extending over the time covered by the radial velocity observations of which I am aware is that by Osthoff,‡ from 1908 to 1922. Osthoff compared α Orionis with both β Orionis and α Tauri. The two series of com-

* *M.N.R.A.S.*, **81**, 701, 1911.
† *Popular Astronomy*, **21**, 5, 1913.
‡ *A.N.*, **216**, 187, 1922.

parisons agree well with one another. Maximum brightness occurred approximately at 1908·0, 1914·0, and 1920·0; minimum brightness occurred at 1911·0, 1917·0. Maximum and minimum radial velocities occur at phases $0^{y}\cdot84$ and $4^{y}\cdot46$ according to the elements derived above. Therefore maximum velocities occurred at 1911·6, 1917·4, and minimum velocities at 1909·4, 1915·2, 1921·0. It appears therefore that the velocity and light variations are correlated, the maximum velocity of recession coming about six months after minimum brightness and maximum velocity of approach coming about one year after maximum brightness.

The magnitude estimates of Osthoff are supported by comparisons made by Barnard between 1918 and 1923, quoted by Pease.* Barnard's observations also bring into evidence the existence of the short-period fluctuations in brightness.

A well-defined maximum of α Orionis occurred in 1902 October and was observed by a number of observers, both visually and photographically.† The radial velocity was a minimum at 1903·65, which again follows maximum brightness by approximately one year.

The correlation between radial velocity and brightness for α Orionis is therefore similar to that found for many Cepheid variables, for which maximum light usually occurs slightly before maximum velocity of approach.

The available evidence seems strongly to support the view that there is a close correlation between the velocity and light changes of long period. If this is always the case, the old observations of brightness will provide evidence on the point previously referred to—whether or not the long-period variation is a regular one. The observations by Herschel and Baxandall, reduced by Plummer,‡ appear to indicate that both period and amplitude of variation are irregular to some extent, though the observations are not sufficiently numerous to enable the times of maxima and minima to be accurately fixed, particularly as the short-period variations are superposed. It is therefore probable that the main radial velocity variation has also variable period and amplitude.

α Scorpii.

The observations of α Scorpii, as published in the *Cape Annals*, **10,** and the *Lick Observatory Publications*, **16,** are summarised in Table IV., which is arranged in a manner analogous to Table I. The observations are not so numerous as in the case of α Orionis; they comprise 101 determinations at the Cape and 49 at Lick. In order to bring the measures at the two observatories into the best general agreement, a correction of − 0·26 km./sec. has been applied to the published Lick velocities. This correction is incorporated in the mean velocities given in Table IV.

* *P.A.S.P.*, **34**, 346, 1922.
† *Geschichte und Literatur des Lichtwechsels*, Bd. I., p. 173.
‡ *Loc. cit. ante.*

TABLE IV.

Radial Velocity Observations of α Scorpii.

Cape.			Lick.		
Mean Date.	Mean Velocity.	No. of Obs.	Mean Date.	Mean Velocity.	No. of Obs.
	km./sec.			km./sec.	
1904·60	−4·60	1	1897·39	−5·27	3
06·18	0·82	13	99·55	1·50	1
06·66	1·86	12	1902·40	4·70	1
07·18	2·63	8	04·48	2·80	1
07·63	2·75	12	05·29	0·04	5
08·19	3·62	5	06·34	1·75	2
08·61	5·45	6	07·41	3·30	1
09·57	3·97	7	08·19	5·10	1
10·15	4·90	2	09·43	3·38	6
10·68	3·83	6	10·27	1·86	4
11·11	4·80	6	10·54	4·87	3
11·64	3·46	5	11·22	4·05	2
12·17	3·37	10	11·62	4·04	5
12·65	−1·99	8	17·14	4·83	3
			17·52	1·73	3
			18·51	5·65	2
			20·34	0·22	4
			22·37	−5·20	2

The observations were plotted graphically and a period of 7·25 years was adopted as a provisional value for the period. With this period all the observations were reduced to one revolution and a smooth curve drawn to represent them as closely as possible. From this curve preliminary elements were derived by the method of Lehmann-Filhés. The elements were as follows :—

$$K = 1{\cdot}92 \text{ km./sec.}$$
$$V_0 = -3{\cdot}23 \text{ km./sec.}$$
$$e = 0{\cdot}48$$
$$\omega = 321^\circ$$
$$T = 1905{\cdot}50$$
$$P = 7{\cdot}25 \text{ years.}$$

The Lick and Cape observations were combined into 28 normal places, arranged in order of date, details of which are given in Table V. The velocity residuals which result from the above elements were computed and normal equations formed, as in the case of α Orionis,

TABLE V.

Observations of α Scorpii: Normal Places.

Date.	Mean Observed Velocity.	No. of Plates.	Revolution.	Final Phase.	Computed Velocity.	δV (O−C).	Weight.
	km./sec.			y	km./sec.	km./sec.	
1897·39	−5·53	3	−2	6·55	−3·50	−2·03	0·6
99·55	1·76	1	−1	1·36	2·41	+0·65	0·2
1902·40	4·96	1	,,	4·21	4·27	−0·69	0·2
04·48	3·06	1	,,	6·29	3·88	+0·82	0·2
04·60	4·60	1	,,	6·41	3·71	−0·89	0·2
05·29	0·30	5	,,	7·10	1·53	+1·23	1·0
06·18	0·82	13	0	0·64	1·01	+0·19	2·6
06·34	2·01	2	,,	0·80	1·38	−0·63	0·4
06·66	1·86	12	,,	1·12	2·02	+0·16	2·4
07·18	2·63	8	,,	1·64	2·77	+0·14	1·6
07·41	3·56	1	,,	1·87	3·02	−0·54	0·2
07·63	2·75	12	,,	2·09	3·22	+0·47	2·4
08·19	3·91	6	,,	2·65	3·63	−0·28	1·2
08·61	5·45	6	,,	3·07	3·86	−1·59	1·2
09·43	3·64	6	,,	3·89	4·18	+0·54	1·2
10·15	4·90	2	,,	4·61	4·35	−0·55	0·4
10·27	2·12	4	,,	4·73	4·37	+2·25	0·8
10·54	5·13	3	,,	5·00	4·39	−0·74	0·6
10·68	3·83	6	,,	5·14	4·39	+0·56	1·2
11·11	4·80	6	,,	5·57	4·34	−0·46	1·2
11·22	4·31	2	,,	5·68	4·31	0·00	0·4
11·63	3·88	10	,,	6·09	4·09	+0·21	2·0
12·17	3·37	10	,,	6·63	3·28	−0·09	2·0
12·65	1·99	8	,,	7·11	1·49	−0·50	1·6
17·14	5·09	3	+1	4·25	4·28	−0·81	0·6
17·85	3·56	5	,,	4·96	4·39	+0·83	1·0
20·34	0·48	4	+2	0·10	0·27	−0·21	0·8
22·37	−5·46	2	,,	2·13	−3·26	−2·20	0·4

for the correction of the orbital elements. The normal equations were :—

δG	δK	π	ε	τ	m	
+ 28·600	− 8·461	+ 6·906	+ 10·055	− 0·572	− 0·914	= + 1·40
	+ 13·524	− 0·076	+ 0·098	+ 0·437	− 0·125	= + 0·45
		+ 15·076	+ 2·273	+ 5·656	+ 1·797	= − 0·60
			+ 6·046	− 0·287	− 0·570	= + 0·74
				+ 3·161	+ 1·171	= − 0·17
					+ 0·885	= + 0·19.

In these equations unit weight corresponds to the mean of 5 plates and the unit of time used in the formation of the coefficients of m is 10 years.

The solution of these equations gives :—

$$\begin{aligned} \delta G &= +0{\cdot}215 \pm 0{\cdot}13 \\ \delta K &= +0{\cdot}166 \pm 0{\cdot}15 \\ \pi &= -0{\cdot}305 \pm 0{\cdot}23 \\ \epsilon &= -0{\cdot}041 \pm 0{\cdot}30 \\ \tau &= +0{\cdot}226 \pm 0{\cdot}53 \\ m &= +0{\cdot}755 \pm 0{\cdot}58 \end{aligned} \qquad \begin{aligned} \delta\varpi &= +9{\cdot}1^\circ \pm 7{\cdot}1^\circ \\ \delta e &= +0{\cdot}007 \pm 0{\cdot}054 \\ \delta T &= +0{\cdot}042 \pm 0{\cdot}100 \\ \delta P &= +0{\cdot}101 \pm 0{\cdot}078. \end{aligned}$$

The corrections are of the same order as the probable errors of the determination, and a second approximation was not considered necessary.

The following were therefore adopted as the elements which provide the best representation of the Cape and Lick observations. Halm's elements are given alongside for comparison :—

	New Elements.	Halm's Elements.
G	$-2{\cdot}30 \pm 0{\cdot}13$ km./sec.	
K	$2{\cdot}09 \pm 0{\cdot}15$ km./sec.	2·12 km./sec.
e	$0{\cdot}487 \pm 0{\cdot}054$	0·20
ω	$330^\circ{\cdot}1 \pm 7^\circ{\cdot}1$	289°
T	$1905{\cdot}54 \pm 0{\cdot}10$	1904·53
P	$7{\cdot}351 \pm 0{\cdot}078$ years	5·80 years
V_0	$-3{\cdot}18 \pm 0{\cdot}18$ km./sec.	−3·09 km./sec.

The new elements differ widely from those derived by Halm.

The probable errors assigned to the elements have been based upon a mean probable error for a single plate (Lick and Cape plates receiving the same weight) of $\pm 0{\cdot}83$ km./sec. This value was derived from the residuals for each individual plate, resulting from the above elements. The representation of the observations by these elements is shown in fig. 2, in which the observations have been grouped according to phase, irrespective of the revolution number.

The values of the projected semi-axis major and of the mass-function are :—

$$a \sin i = (67{\cdot}4 \pm 4{\cdot}8) \times 10^6 \text{ kms.}$$
$$f = 0{\cdot}00170 \odot.$$

The residuals for the normal places are contained in the second last column of Table V. As in the case of α Orionis, there is no indication of a change in the period or in the amplitude of the velocity variation during the interval over which the observations extend. It must be remembered, however, that this interval is only 25 years and is less than four times the period ; the evidence therefore does not conclusively establish the constancy of either the period or the amplitude. It is again evident from an examination of the residuals, considered in conjunction with the number of plates upon which each depends, that

short-period oscillations are present. This conclusion is supported by the probable errors. From the residuals in the last column of Table V. the probable error corresponding to unit weight (the mean of 5 plates) is found to be ± 0·62 km./sec. The probable error for a

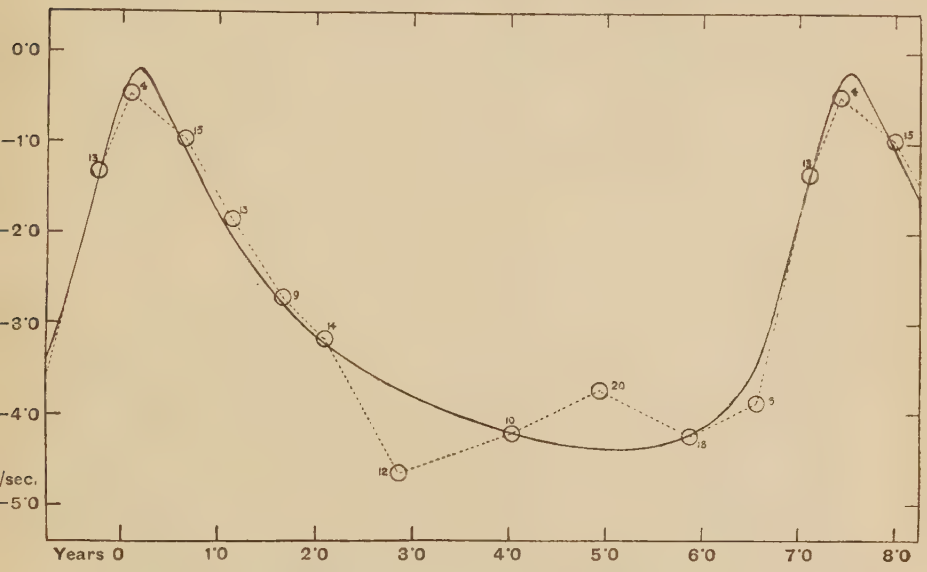


Fig. 2.—The Radial Velocity Curve of α Scorpii.

single plate is therefore ± 1·39 km./sec., whereas the value obtained from the individual residuals, as mentioned above, is ± 0·83 km./sec.

To indicate the existence of short-period oscillations, some of the larger residuals in Table V. may be examined in detail, as was done in the case of α Orionis. Table VI. is arranged in a similar manner to Table III.

Table VI.

Cape.		*Cape.*	
Date.	V. km./sec.	Date.	V. km./sec.
1908 July 30	−4·7	1910 Aug. 2	−4·2
Aug. 10	6·0	3	4·5
11	5·4	Sept. 6	4·1
12	5·3	21	3·5
13	5·8	29	3·5
21	5·5	Oct. 4	3·2
P.E. single plate	±0·30	P.E. single plate	±0·34
P.E. of mean	±0·12	P.E. of mean	±0·14
Resid. from curve	−1·59	Resid. from curve	+0·56

TABLE VI.—*continued.*

Cape.		*Lick.*	
Date.	V. km./sec.	Date.	V. km./sec.
1911 Jan. 31	−4·2	1905 Mar. 22	−0·6
Feb. 2	4·8	28	−1·4
6	4·7	29	+0·2
15	5·5	May 14	+1·6
16	4·7	23	0·0
20	4·9		
		P.E. single plate	±0·76
P.E. single plate	±0·28	P.E. of mean	±0·34
P.E. of mean	±0·11	Resid. from curve	+1·23
Resid. from curve	−0·46		

This table shows, as in the case of α Orionis, that where a group of observations secured within a short period shows a large residual from the mean curve, the residual is much larger than would be anticipated from the internal accordance of the group ; in other words, all the plates constituting the group stand out in the same manner. The two groups of Cape plates for 1910 and 1911 in the above table are of interest : the radial velocities as given by the adopted elements differ by only 0·05 for the mean epochs of the two groups, yet the means of the two groups—each of which shows an excellent internal agreement—differ by 1·0 km./sec. The largest residual in Table V. is + 2·25 km./sec. for a group of four Lick plates : these give velocities of − 1·0, − 3·0, − 2·9, − 0·5 km./sec., whereas the computed velocity is − 4·4 km./sec. Not one of the four plates comes within 1·4 km./sec. of this value.

The magnitude of the short-period fluctuations may be estimated in the same way as for α Orionis. The Cape and Lick plates will again be treated separately. From the residuals of the two series, the total probable error of a single plate (including the errors which arise from the short-period fluctuations) is found to be ± 0·71 km./sec. for Cape plates and ± 1·04 km./sec. for Lick plates. These values correspond very closely to those found for α Orionis. If we take, as before, differences of the residuals of consecutive plates obtained at the same epoch, the probable error of a difference is found to be ± 0·68 km./sec. for Cape plates and ± 1·00 km./sec. for Lick plates.

If, then, we define σ_C, σ_L and σ as before, we have :

$$\sigma_C^2 + \sigma^2 = 0{\cdot}50$$
$$\sigma_L^2 + \sigma^2 = 1{\cdot}08$$
$$2\sigma_C^2 = 0{\cdot}46$$
$$2\sigma_L^2 = 1{\cdot}00$$

whence

$$\sigma_C^2 = 0{\cdot}30, \quad \sigma_L^2 = 0{\cdot}53, \quad \sigma^2 = 0{\cdot}42.$$

We therefore conclude that :

Probable error of one Cape plate	= ± 0·45	km./sec.
,, ,, ,, Lick ,,	= ± 0·73	,,
,, ,, due to fluctuations	= ± 0·65	,,

These errors are of the same order of magnitude as the corresponding errors for α Orionis. In particular, the probable error due to the short-period fluctuations is ± 0·65 for α Scorpii as compared with ± 0·68 for α Orionis, almost exactly equal in magnitude.

The velocity changes of α Orionis were correlated with the changes of brightness. In the case of α Scorpii, however, there is no correlation as the light of the star is not variable.

The Nature of the Radial Velocity Variations.

Orbital elements have been derived above for the velocity variations of α Orionis and α Scorpii as though the stars were spectroscopic binaries. This does not necessarily imply that they are actually binary systems. If the velocity variations are attributed to pulsations in these stars, the values of $a \sin i$ are to be interpreted as giving the distance through which the star moves on each side of its mean position. The derived values of $a \sin i$ may be compared with the radii of the two stars, the radii deduced from theoretical considerations having been satisfactorily confirmed by measures with the 20-foot stellar interferometer at the Mount Wilson Observatory.

The measured angular diameter of α Scorpii is 0″·040 which, assuming a parallax of 0″·0095, corresponds to a radius 450 times that of the Sun. The angular diameter of α Orionis, as measured with the interferometer, has ranged from 0″·034 to 0″·047. The individual values, deduced from the separations given in the *Mount Wilson Annual Reports*, 1921–1927, are as follows :—

	″
1921	0·047
1922	·055
1923	·034
1924	·041
1925	·044
1926	·034
1927	·041.

The epochs of observation are generally near the beginning of the years given in the first column. Adopting 0″·042 as the mean diameter, the radius of the star on the assumption of a parallax of 0″·017 is 270 times that of the Sun. We have therefore the following comparison :—

	$a \sin i$.		R.	$a \sin i$/R.
	10^6 kms.	⊙'s radii.	⊙'s radii.	
α Orionis	58·5	84	270	0·31
α Scorpii	67·4	97	450	0·22

If the velocity variations are interpreted as pulsations, the semi-amplitude of the variation in the radius of the star is 0·31 of the mean radius for α Orionis and 0·22 of the mean radius for α Scorpii. These values would need to be increased somewhat to allow for the fact that parts of the surface of the star are moving obliquely to the line of sight. Russell estimates that an increase of about 40 per cent. to the computed values of $a \sin i$ should be made to allow for this ; the ratios, $a \sin i/\mathrm{R}$, would then be increased to 0·43 and 0·31. It may be noted that the extreme range of the diameters of α Orionis observed at Mount Wilson, as given above, is 0″·021 or exactly half the mean diameter.

The ratio, $a \sin i/\mathrm{R}$, is smaller for α Scorpii than for α Orionis, but the changes in angular diameter should be observable with the 20-foot interferometer at Mount Wilson. It is surprising that no variations have been detected. It is possible that variations will be revealed by the more accurate observations which will be possible on completion of the new 50-foot interferometer.

For the normal Cepheid variable, considered as a pulsating star, the semi-amplitude of the variation in the radius is of the order of 8 or 10 per cent. of the radius. For the long-period variable, Mira Ceti, the ratio $a \sin i/\mathrm{R}$ is found to be 0·13, corresponding to a semi-amplitude of the variation in radius of the order of 18 per cent. of the radius. The pulsations of α Orionis and of α Scorpii are larger than for any other known stars.

According to Eddington's theory of a pulsating star, there is a limiting amplitude of the pulsation which cannot be exceeded. Eddington * gives as this limit a value of $\delta\mathrm{R}/\mathrm{R}$ between $\frac{1}{5}$ and $\frac{1}{6}$. The values for α Orionis and α Scorpii are appreciably higher than this. Eddington's equation connecting the pressure change and $\delta\mathrm{R}/\mathrm{R}$ is †

$$\mathrm{P}_1 = -(4 + 12{\cdot}2\,\gamma\alpha)\frac{\delta\mathrm{R}}{\mathrm{R}},$$

where

$$\mathrm{P} - \mathrm{P}_0 = \delta\mathrm{P} = \mathrm{P}_0\mathrm{P}_1$$

and

$$\alpha = 3 - \frac{4}{\gamma}.$$

Also the theory leads to the equation ‡

$$\Pi\sqrt{\rho_c} = 0{\cdot}290(\gamma\alpha)^{-\frac{1}{2}}$$

where Π is the period of variation and ρ_c is the central density.

Therefore

$$\mathrm{P}_1 = -\left(4 + \frac{c}{\Pi^2\rho_c}\right)\frac{\delta\mathrm{R}}{\mathrm{R}},$$

where c is an absolute constant. Since the pressure cannot become

* Eddington, *Internal Constitution of the Stars*, § 131, p. 194.
† *Loc. cit.*, Equations (131.2) and (131.3).
‡ *Loc. cit.*, Equation (130.4).

negative, $-P_1$ cannot be greater than unity. The value of $\frac{\delta R}{R}$ cannot therefore exceed

$$1/\left(4+\frac{c}{\Pi^2\rho_c}\right).$$

For the normal Cepheid variables, the values of $\Pi\sqrt{\rho_c}$ average about 0·8, with an extreme value of about 1·4. For α Orionis and α Scorpii the values are much greater. The mean densities of these stars may be taken as 6×10^{-7} and 3×10^{-7} respectively in gm./cm.^{3} units. Adopting Eddington's value of $\rho_c/\rho_{mean}=54{\cdot}36$, the values of $\Pi\sqrt{\rho_c}$ for the two stars are as follows :—

$$\begin{aligned}\alpha\text{ Orionis }\ \Pi\sqrt{\rho_c} &= 12{\cdot}1\\ \alpha\text{ Scorpii} &= 10{\cdot}8,\end{aligned}$$

or about fifteen times as large as for a normal Cepheid. The term $\frac{c}{\Pi^2\rho_c}$ has therefore for these stars a value only about the $\frac{1}{2\,2\,5}$th part of its value for an average Cepheid. The value of δR/R can therefore amount to practically 0·25. The observed values are somewhat greater than this. With pulsations of this magnitude, however, it is necessary to take into account second order terms which have been neglected in the theory. It is possible that their inclusion may increase the maximum possible amplitude of pulsation to the values found for the two stars under consideration.

The large value for $\Pi\sqrt{\rho_c}$ has another consequence. It implies that the quantity $(\gamma\alpha)$ for these two stars is much smaller than for the average Cepheid, and therefore that the ratio of the specific heats of the matter of which they are constituted approaches more nearly to the value $\frac{4}{3}$ than in the case of the average Cepheid.

A correlation between the radial velocity variations and the variations of angular diameter of α Orionis would be anticipated and was suspected by Pease * from the early observations. Later observations, as given above, make the correlation more uncertain. It may be of significance that the greatest angular diameter so far measured was near 1922·0. The radial velocity was a minimum at 1921·0 and a maximum at 1923·2. The greatest angular diameter occurred therefore after maximum velocity of approach and approximately at the mid-interval between maximum velocity of approach and maximum velocity of recession. The star had ceased to expand and was about to commence contracting. A maximum angular diameter would therefore be expected about 1922·0. The minimum diameter of 0″·034 in 1926 follows maximum velocity of recession and precedes maximum velocity of approach and is again in accordance with expectation. The value of 0″·034 obtained in 1923 is not in accordance, however, with the value to be expected from the radial velocities. The observation may have been in error, but evidently further observations will be required to establish a definite correlation between the changes in diameter and in radial velocity.

* *P.A.S.P.*, **34**, 34*b*, 1922.

The following relation connecting R, the radius of a star (in terms of the Sun's radius as unity), M_v, the absolute visual magnitude, and I, the colour index, is given by Russell * :—

$$\log R = 0{\cdot}82I - 0{\cdot}20M_v + 0{\cdot}51,$$

whence

$$\log\left(1 + \frac{\delta R}{R}\right) = 0{\cdot}82\delta I - 0{\cdot}20\delta M_v.$$

In the case of Betelgeuse, the total amplitude of the principal light variation is about $0^m{\cdot}7$, so that the extreme values of δM_v are $\pm 0{\cdot}35$. Due to change of brightness alone, there is therefore a change in colour index of the order of $\pm 0^m{\cdot}09$, the star being less red than normal at maximum brightness and redder than normal at minimum brightness. A variation of this nature is typical of Cepheids. We have seen also that the extreme values of $\delta R/R$ (if a correction of 40 per cent. is applied to allow for the motion of parts of the surface oblique to the line of sight) are $\pm 0{\cdot}43$. The resulting changes in colour index, due to the variation in R, are $+ 0^m{\cdot}19$ and $- 0^m{\cdot}30$, the star being more red than normal at maximum expansion and less red than normal at maximum contraction. This effect predominates over that due to change in brightness. The combined effect of the changes in radius and brightness is practically that the star is redder than normal on the upward slope of the velocity curve and less red than normal on the downward slope of the curve. There do not appear to be any reliable determinations of the colour changes of α Orionis and of their correlation with the magnitude changes.

No perceptible changes of brightness occur in the case of Antares, so that δM_v is zero. The changes in radius due to the pulsations should produce corresponding changes in the colour. There is no record of such changes having been observed. This is not conclusive evidence against their occurrence, as changes of colour in a bright red star such as α Scorpii may easily escape notice. If such changes occur, the photographic magnitude of the star will be variable, although the visual magnitude is constant or approximately so.

Summary.

1. The radial velocity changes of α Orionis and of α Scorpii have been investigated. The orbital elements are :—

	α Orionis.	α Scorpii.
V_0 (km./sec.)	+ 21·05 ± 0·15	− 3·18 ± 0·18
K („)	2·06 ± 0·09	2·09 ± 0·15
e	0·208± 0·039	0·487± 0·054
ω	285°·8 ±11°·1	330°·1 ±7°·1
T	1904·97 ± $0^y{\cdot}18$	1905·54±$0^y{\cdot}10$
P (years)	5·781± 0·040	7·351± 0·078
$a \sin i$ (10^6 kms.)	58·5 ± 2·6	67·4 ± 4·8

* *Vide* Russell, Dugan, and Stewart, *Astronomy*, p. 738.

2. Superposed on the main variation are irregular short-period variations, with periods of the order of a few months.

3. The radial velocity changes of α Orionis are correlated with the changes of brightness in a manner similar to the changes for Cepheid variables. The radial velocity variation is therefore probably variable both in period and amplitude.

4. A correlation between the radial velocity changes and the changes in diameter of α Orionis is suspected but cannot at present be considered as definitely established.

5. The ratio of $a \sin i/R$ is 0·31 for α Orionis and 0·22 for α Scorpii. The large amplitude of the variation in the radius is related to the large value of Period $\times$ (Central density)$^{\frac{1}{2}}$, and both are due to the ratio of the specific heats of the matter of which the star is constituted approaching more nearly to the value $\frac{4}{3}$ than in the case of the average Cepheid.

6. Both α Orionis and α Scorpii should show changes in colour which should be correlated with the radial velocity changes. Little or nothing is known about the colour changes of either star.

Royal Observatory, Cape of Good Hope :
1928 *May* 10.

On the Presence of Radiation Bands of Aluminium Oxide in the Spectrum of Mira Ceti at the Maximum of 1924 *February.* By F. E. Baxandall, A.R.C.Sc. (Plate 12.)

(*Communicated by the Director, Solar Physics Observatory, Cambridge.*)

In a comprehensive paper on "A Spectroscopic Study of Mira Ceti,"* Alfred H. Joy points out on p. 327 that "the exceptionally low maximum of 1924 February is especially interesting because of the appearance at that time of a new and unknown series of bright lines or bands and the presence of broad absorption bands in the region λ 4600 to λ 4900, which have not hitherto been noted in any stellar spectrum."

In Plate xx. of his paper, the spectrum at this maximum (mag. 4·7) is shown alongside a spectrum taken in 1923 March (mag. 3·2), the latter representing the usual spectrum at or near a maximum.

Joy also says, "The most prominent band, after correction for the usual velocity of bright lines, has its centre at λ 4844·7. Its width is about 7 A. Other strong bright regions less than 1 A. in width are found at $\lambda\lambda$ 4535·5, 4649·15, 4672·76, 4695·40, and 4867·53. These have the appearance of rather wide bright lines, but must be related to the accompanying bands, for there seems to be an absorption band from 5 to 15 A. in width to the violet of each of the bright regions. The

* *Ap. J.*, **63**, 281–341.

absorption . . . seems to fade towards the violet, opposite to the titanium bands."

It seemed to the writer that as the new bands are either radiation bands fading to red or absorption bands fading to violet, the effective wave-lengths to be used *in a search for the origin of the bands* should be those of the " junction " of the bright and dark band in each case. Joy had recorded the wave-lengths of the middles of the brighter bits of radiation near the head of each band, and also the widths of these measured portions. His wave-lengths have therefore been slightly corrected by deducting the semi-width of each line from the recorded position, giving the following wave-lengths :—

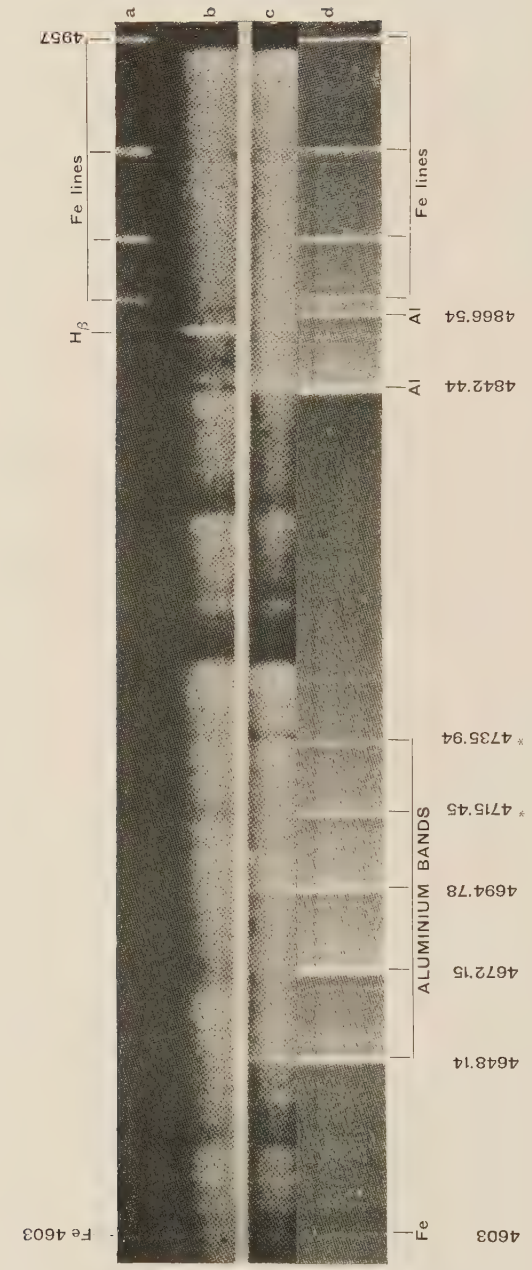
4535·0, 4648·65, 4672·26, 4694·90, 4841·2, 4867·03.

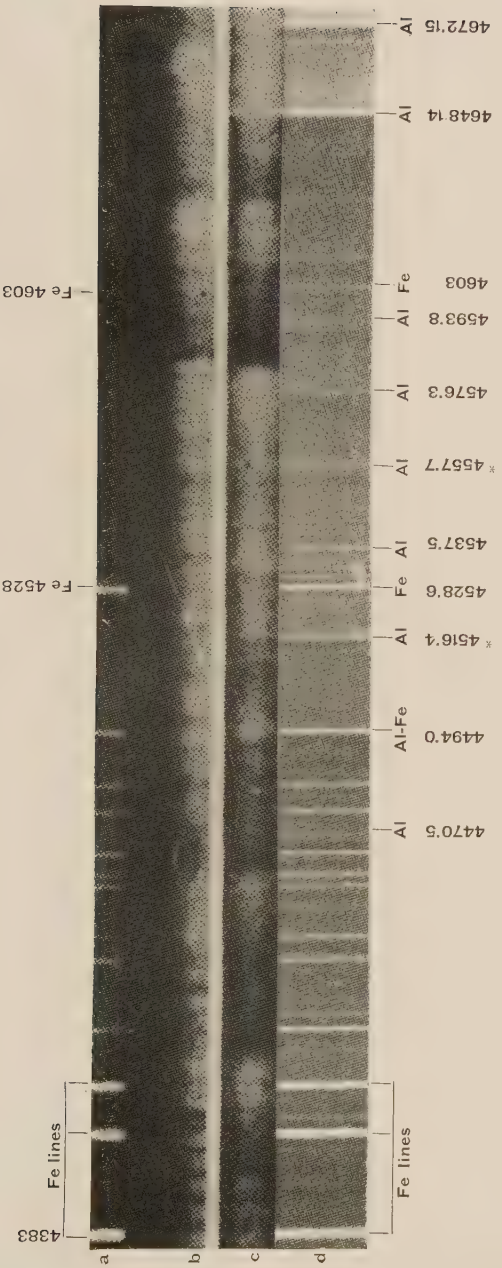
These were used as a basis and reference made to Kayser's *Tabelle der Bandenspectra*. It was seen at once that there is an aluminium band agreeing closely in wave-length with each of the wave-lengths mentioned above. Apart from the first, which shows a difference of 2·5 A., and the fifth, with a difference of slightly over 1 A., the others are well within 1 A. Reference to the published records for aluminium-oxide bands showed that in the range of spectrum covered by Joy's six new lines or bands only eight band-heads occur. The strongest five of these agree with the last five of Joy's six bands. Hasselberg's wave-lengths (corrected to the I.A. scale) for the laboratory bands which show close agreement with Joy's bands are :

4537·52, 4647·96, 4671·97, 4694·60, 4842·26, 4866·36.

Taking into account the wide range of spectrum (330 tenth-metres) over which these small numbers of lines from two different light sources are spread, the succession of close agreements in position very strongly suggested that the stellar and laboratory bands are identical. An enlarged copy of Joy's reproduction of the *o* Ceti spectra was made, and several laboratory spectra of aluminium oxide of varying dispersions were enlarged to the same scale. Some of these did not show similar dispersion to that of the stellar spectra throughout the range represented by the latter and were discarded. One, however, taken by Mr. Bryan Cookson in 1908, showed an almost exact fit throughout. This was placed alongside Joy's spectrum and is shown in the plate accompanying this paper.

As Joy used an iron-comparison on his stellar plate it is rather fortunate that the laboratory spectrum mentioned shows the stronger iron lines as well as aluminium bands. These serve to check the similarity of the dispersion of the laboratory and stellar spectra. Reference to the plate will show that the four iron lines to the right of the upper half are in good agreement in the two spectra, and the lengthy series of iron lines in the left half of the lower set of spectra also show a close fit. The plate also shows that there is a succession of close agreements between the aluminium band-heads and radiation bands or lines in the stellar spectrum. The last five of Joy's six bands whose wave-


a
b
c
d
4957
Fe lines
H_β
Fe lines
Al
4866.54
Al
4842.44
* 4735.94
* 4715.45
4694.78
4672.15
ALUMINIUM BANDS
4648.14
Fe
4603
Fe 4603



(a) Joy's Fe Comparison Spectrum. (b) Spectrum of o Ceti (Max. of 1923 Mar. 28, Joy).
(c) Spectrum of o Ceti (Max. of 1924 Feb. 11, Joy). (d) Cambridge Spectrum of Al Oxide.

Spectra of o Ceti and Aluminium-Oxide Bands.

Recorded Absorption Bands in o Ceti which have been hitherto unidentified.

o Ceti.		Aluminium Oxide Bands.
λ Stebbins.	λ Sidgreaves.	λ Hasselberg.
4713·2	4714	4715·45
..	4736	4735·94
4841·8	4842	4842·44
..	5074	5079·52
..	5098	5102·32
..	5135	5143·08
		Lockyer.
..	5406	5409·0
5438·3	..	5439·5

The third column shows the wave-lengths of aluminium-oxide bands which fall near the same positions as the stellar bands. It is seen that this residuum of bands of hitherto unknown origin has no line for which there is no approximately coincident aluminium band. It may be pointed out here that Sidgreaves' wave-lengths for five stellar bands in the region λ 4950 to λ 5240 ascribed to TiO_2 are consistently low (mean difference about 4 A.). If, then, Sidgreaves' bands of unknown origin in the region mentioned, λλ 5074, 5098, 5135, be corrected for this constant difference of 4 A., two of the resulting wave-lengths are in very close agreement with two aluminium bands, but the third would be still several Angstroms out of agreement.

It thus seems very probable that absorption bands of aluminium oxide exist in the normal spectrum of o Ceti and that *all* the recorded bands other than those of TiO_2 may be ascribed to aluminium. The absence of other aluminium bands from the published records of o Ceti bands is probably explained by the masking effect of the numerous strong and broad TiO_2 bands in the same spectrum. The band 4841·8 in the table is the strongest of the whole series of aluminium bands, and it is this band which was specially referred to in the earlier part of the paper.

There is no spectrum of o Ceti available here which extends to the region of higher wave-length than λ 4900, so it is not possible to make a direct comparison of the stellar and laboratory bands in that region. A more rigorous test of the identity of some of the absorption bands of the normal o Ceti spectrum with aluminium bands can be made by a direct comparison of the stellar and laboratory spectra taken—with fairly large dispersion—throughout the range of spectrum λ 4470 to λ 5450, which includes the chief series of aluminium bands.

Further Reference to Plate.

The top and bottom halves of the plate, which show an overlap in the spectrum from about λ 4600 to λ 4675, are made up from strips photographed from *one* setting on the enlarging screen of *each* of the aluminium spectrum and our photograph of Joy's reproduction of o Ceti. To get

the laboratory and stellar spectra to the same scale, Fe 4957 at the extreme right of the top strip and Fe 4383 at the extreme left of the bottom strip were adjusted to fit in the two spectra. The fitting of the two sets of iron lines (Joy's comparison lines and the "impurity" lines of Fe in the aluminium spectrum) is seen much better in the bottom half of the plate than in the top. The line in the top half marked Fe 4603, however, which is only weak in Joy's comparison spectrum, and is scarcely detectable in the photograph accompanying this paper, could be seen on the enlarging screen to fit exactly the same Fe line in the aluminium spectrum.

Owing to the relative weakness of the series of Al bands from λ 4470 to λ 4594, together with the occurrence of several very strong TiO_2 bands in that region, the identity of these with bands of stellar radiation is not very manifest. The top half of the plate, however, showing coincidences of seven stronger bands of Al in succession with radiation lines or bands in the *o* Ceti spectrum seems to leave little doubt as to the identity.

It is a pleasure to acknowledge the valuable help given by Mr. W. H. Manning in preparing the plate.

Further Observations of some Bright-hydrogen-line Stars, 1924–1927. By William J. S. Lockyer, M.A., Ph.D., Major (late R.A.F.). (Plate 13.)

A continuous watch on the behaviour of the spectra of some of the brighter bright-hydrogen-line stars, not only to investigate their variability but also their possible change in the course of time, is the object of this research. This communication is a further contribution in continuation of my former series of observations * which was confined to the period 1923 September to 1924 February. The present series embraces the period 1924 March to 1928 April.

The original working list has been more or less adhered to, but some minor changes have been made.

The stars have not been observed as continuously as was desired, but this was due to observation being mainly restricted to a concentration on securing the spectra of H.D.C. 10516 (ϕ Persei) and H.D.C. 20336 (– Camelopardalis), two very interesting stars which exhibited marked changes in their spectra, which necessitated photographs being taken as often as possible.

Accounts of the observations of these two stars have already been published, two papers on the former star being communicated to the Society in 1924 and 1925,† and one on the latter in 1926.‡

The total number of photographs obtained in this new survey is 546. Of these, 176 and 4 have already been utilised in the above-mentioned

* *M.N.R.A.S.*, **84**, 409, 1924.
† *Ibid.*, **85**, 580, and **86**, 474.
‡ *Ibid.*, **86**, 607.

discussions on ϕ Persei and H.D.C. 20336 respectively, and 152 are negatives of the latter star, forming a new series (still in progress) for discussion later.

Numerous other negatives of stars which were considered possible to exhibit bright hydrogen lines were taken, but these, however, gave negative results, so they are not included in the above total.

For satisfactory negatives, exposures of about one hour or more are required for ϕ Persei (mag. 4·19), while at least two hours are wanted for H.D.C. 20336 (mag. 4·76).

Up to 1926 October 18 all the photographs, with very few exceptions, were taken by the writer, but since then Mr. D. L. Edwards and the research student Mr. H. W. P. Richards made all the exposures.

All the photographs have been taken with the same instrument as formerly used, namely, a 9-inch prismatic camera having a prism of 45° angle and a dispersion of 46·0 mm. between H_β and H_δ. The photographic plates used throughout have been Ilford's "Iso-Zenith," with a speed number of H and D 700, and the developer employed Burroughs Wellcombe & Co.'s tabloid "Rytol."

It may be mentioned that the bright-hydrogen-line stars are mostly confined to the spectral type B3. The general appearance of their spectra is marked by broad absorption bands of hydrogen, on which are generally seen double bright sharp components symmetrically placed on them which may or may not vary their relative intensities. The helium absorption lines are mostly broad, strong, and nebulous. The ionised magnesium line at λ 4481·3 is usually rather weak and nebulous, and the ionised absorption line of calcium (K) at λ 3933·8 moderate to weak in intensity and variable as regards sharpness.

Between H_β and H_γ there sometimes appear numerous double bright sharp ionised lines chiefly of the metals iron and titanium, the same as those that are most prominent in the spectrum of ϕ Persei.*

The ionised absorption lines at $\lambda\lambda$ 4471·7 and 4481·3, which will often be mentioned later in the descriptions of each star, will for brevity be referred to only as the helium and magnesium lines with their wavelengths, and the mention of the ionised absorption line of calcium at λ 3933·8 as "K" only.

Apart from the changes which have been noted as taking place in the spectrum of H.D.C. 20336, the spectrum of which is still under close observation, the new series of observations has led to quite distinct variations in the appearance of H_β in the spectrum of H.D.C. 37202 (ζ Tauri). A very probable alteration in the spectrum of H.D.C. 23480 (23 Tauri, Merope) has also occurred. A star more recently added to the working list is H.D.C. 58715 (β Canis Minoris), and this exhibits a change in spectrum by comparison with descriptions given by earlier observers.

Most of the other stars examined do not seem to have varied their spectra so far as can be definitely stated.

In the following brief descriptions of each of the stars placed in order of right ascension, the heading includes the Henry Draper Catalogue

* For a list of these lines see *M.N.R.A.S.*, **85**, 590, 1925.

number, the name of the star, Right Ascension and Declination for 1900, the photometric magnitude, all taken from the same catalogue, and the spectrum type. The letters following the spectrum type are given according to the notation adopted by the International Astronomical Union. The " n " means that the lines (in this spectrum type chiefly helium lines) are " nebulous " ; " e " that bright lines occur in the spectrum ; and " α," " β," or " γ," the suffix for the limiting hydrogen line on which a bright line (or lines) has been recorded in the present paper.

Following the above details, the commencing statement " Last photograph " refers to the date of the last photograph discussed in my previous communication on these stars. The dates following this are those of the first and last negatives taken for each particular star during the present series.

References to " Moore," followed by a number in brackets representing the page, are taken from the *Lick Observatory Bulletin*, No. 355, 1924, " Third Catalogue of Spectroscopic Binary Stars," by J. H. Moore. In this Moore gives three tables, in which Table I. lists all stars for which a variation in radial velocity appears to have been established. Table II. gives those which have been announced as spectroscopic binaries, but whose variable velocity has later been disproved, or at least has been found to depend upon insufficient or doubtful evidence. Lastly, Table III. contains the orbital elements of all spectroscopic binaries for which well-determined or approximate orbits exist. These are the tables to which reference is made under many of the descriptions which follow.

The references to " Merrill," also followed by a page number, are taken from *Contributions from the Mount Wilson Observatory*, No. 294, 1925,* containing a paper entitled " Discovery and Observations of Stars of Class Be," by Paul W. Merrill, Milton L. Humason, and Cora G. Burwell.

They refer to special observations on the intensity of the bright H_α line on various dates, and are helpful in estimating approximately the expected brightness of the H_β, H_γ, etc., lines on the Sidmouth photographs on which the H_α region is not recorded.

In the present communication all references that related to earlier observations on these stars by other observers, many of which were given in my previous paper, have here been excluded.

A plate is added (Plate 13) illustrating some of the salient features of four of the stellar spectra to which reference is made. The broadening of the spectra has been accomplished by means of the clepsydra apparatus.

H.D.C. 5394 (γ Cas.) $0^h\ 50^m{\cdot}7 + 60°\ 11'$ Mag. 2·25 Sp. Boneθ.

Last photograph 1923 December 23. Ten other negatives have since been secured between the dates 1924 July 16 and 1927 October 10, one or more have been taken in each year. They are all of excellent quality.

* Reprinted from *Ap. J.*, **61**, 389, 1925.

In these spectra the bright components of H_β form an exceedingly close double, but in the cases of H_γ, H_δ, H_ϵ, etc., they are well separated, the separation increasing as the violet is approached. In every instance the bright components of each of the hydrogen lines are nearly of equal intensity, that of the violet being perhaps slightly the stronger. There is suspicion, however, in the last negative which was taken that the red components have slightly increased in intensity.

There is no record by any observers that this spectrum has ever undergone any change. It is quite possible, however, that there may be occurring small variations in the intensities or relative intensities of the bright components of the hydrogen lines which have escaped observation owing to the lack of sufficient dispersion in the spectroscopic apparatus employed.

The dispersion of the Sidmouth instrument is quite sufficient to detect any such possible changes, and it is for this reason that this star has been and is being kept under observation.

Moore (163) places this star in his Table II.

Merrill (11) describes the bright H_a as intensity 4 (max. = 5), from four negatives taken between 1919 September 30 and 1921 September 24.

H.D.C. 10516 (ϕ Per.) $1^h\ 37^m{\cdot}4 + 50^\circ\ 11'$ Mag. 4·19 Sp. Boneζ.

Last photograph 1924 February 14. Since then 175 further negatives have been secured, and their discussion has been embodied in two papers entitled "The Spectrum of ϕ Persei (Sp. Type Bope)," *M.N.R.A.S.*, **85**, 580, 1925, and "The Spectrum of ϕ Persei (Sp. Type Bope) throughout a complete cycle 1925–1926," *M.N.R.A.S.*, **86**, 474, 1926.

Moore (166) gives the orbital elements of this star in his Table III.

Merrill (11) estimates the intensity of bright H_a as 4 on a scale of 5 as maximum on 1920 September 20.

H.D.C. 20336 (– Cam.) $3^h\ 11^m{\cdot}2 + 65^\circ\ 17'$ Mag. 4·76 Sp. B3neη.

Last photograph 1924 January 27. Since then 156 photographs have been taken between the dates 1924 October 23 and 1928 April 9.

A discussion of all the photographs taken at Sidmouth up to 1925 December 17 has been published in a paper entitled "The Spectrum of the Bright-hydrogen-line Star H.D.C. 20336 in Camelopardalis (Sp. Type B3pe)," *M.N.R.A.S.*, **86**, 607, 1926.

In order to study the behaviour of the spectrum throughout a complete cycle a new series of photographs was commenced on 1926 June 19, and is still in operation. Up to the present (1928 April 20) 152 negatives have been obtained, and these will be discussed as soon as the series is completed.

Moore (145) includes this star in his Table I.

Merrill (11) gives the intensity of bright H_a as 3 (max. = 5), from three negatives taken between 1919 October 1 and 1920 October 25.

H.D.C. 22192 (ψ Per.) $3^h\ 29^m{\cdot}4 + 47°\ 51'$ Mag. 4·26 Sp. B5neϵ.

Last photograph 1924 February 14. Since then 26 further plates have been secured between the dates 1924 September 2 and 1927 October 6, the number of plates taken in the separate years being 7, 12, 5, and 2 respectively.

The relative intensities of the bright components of the hydrogen lines appear to be still the same, the violet components being a little stronger than the red components.

The helium lines remain nebulous and fairly strong, and that at λ 4471·7 is stronger than the nebulous line of magnesium at λ 4481·3. " K " is sharp and fairly conspicuous, and is the only line which has shown any suspicion of change of intensity. The double bright ionised lines of the metals, the same as those in ϕ Persei, are very sharp and conspicuous in all good negatives.

A rediscussion of the plates examined in my previous paper on this star fails to corroborate the statement there made, that the negative No. 436 appeared to differ from the others. They all show that the violet component of H_β is a little stronger than the red component, as is the case in the new series of plates.

The photographs indicate therefore that no detectable change has taken place in the spectrum.

Moore does not include this star in his tables.

Merrill (11) gives the intensity of bright H_α as 4 (max. = 5), from four negatives taken between 1919 September 30 and 1921 March 27.

H.D.C. 23302 (17 Tau. Electra) $3^h\ 39^m{\cdot}0 + 23°\ 48'$ Mag. 3·81 Sp. B5peα.

Last photograph 1924 February 1. Since then 17 more photographs have been secured between the dates 1924 September 2 and 1927 October 7. In no instances have the hydrogen lines exhibited bright components, but H_β and H_γ at times have indicated narrow absorption lines centrally placed on the broad absorption bands. The helium line λ 4471·7 has always been equal to or possibly a little stronger than the magnesium line at λ 4481·3. " K " has been very weak or nearly absent. It appears that the bright lines previously recorded have now disappeared.

Moore does not mention this star in his tables.

Merrill (16) mentions that H_α was " weak bright on absorption " on 1919 November 8.

H.D.C. 23480 (23 Tau. Merope) $3^h\ 40^m{\cdot}4 + 23°\ 39'$ Mag. 4·25 Sp. B5peγ.

Last photograph 1924 February 1. Since then 17 more negatives have been secured, extending from 1924 September 2 to 1927 October 7. In the spectrum of this star only the H_β and H_γ lines exhibit bright components. H_δ and H_ϵ show narrow absorption lines on broad

absorption bands, and the other hydrogen lines as broad absorption bands only. The helium line at λ 4471·7 is always a little stronger than the magnesium line at λ 4481·3, and "K" is nearly always rather weak.

The bright components of H_β and H_γ seem to have become more strong since the earlier photographs were taken, and in the months of September and October of 1925 the violet component was much stronger than the red component. Further, in 1927 October they became more on an equality. There is ground for believing therefore that this spectrum is variable.

Moore does not include this star in either of his Tables I. or II.

Merrill (16) does not state a value of the intensity of the bright H_α, but mentions that it was "very weak bright on absorption on 1919 November 8."

Plate 13, spectrum number 6, gives an illustration of the spectrum of this star on 1925 October 15 showing H_β as distinctly double bright on broad absorption.

H.D.C. 23630 (η Tau. Alcyone) $3^h\ 41^m{\cdot}5 + 23°\ 48'$ Mag. 2·96
Sp. B5neγ.

Last photographs 1924 February 1. Since then 17 negatives have been secured between the dates 1924 September 2 and 1927 October 7. In the spectrum of this star only the hydrogen lines H_β and H_γ exhibit bright components, H_δ and H_ϵ showing as narrow absorption lines on broad absorption bands, and the rest of the series as broad absorption bands. The helium line at λ 4471·7 is always about equal in intensity to the magnesium line at λ 4481·3, and "K" is generally very weak or absent.

The bright components of H_β are conspicuous in all photographs, and the violet component is on the average a little stronger than the red component, although the former has sometimes been estimated as equal to and even a little weaker than the latter. In none of the spectra do the double bright ionised lines as seen in ϕ Persei make their appearance.

The new photographs are thus very similar to the earlier set, but differ from the photographs taken by Adams in 1903, as previously pointed out, in which he showed that there was no evidence of the presence of a bright line at H_β.

This star does not appear in Moore's Tables.

Merrill (11) gives the intensity of H_α as 1 on a scale of 5 as a maximum, from three negatives taken between 1919 July 1 and 1921 November 1, but also states on another page (16) that H_α was "strong bright" on 1919 November 8.

There is reason to believe that this spectrum is variable.

H.D.C. 23850 (27 Tau. Atlas) $3^h\ 43^m{\cdot}2 + 23°\ 45'$ Mag. 3·80 Sp. B8ne.

Last photograph 1924 February 1. Since then 17 negatives have been secured between the dates 1924 September 2 and 1927 October 7. There seems to be no indication of any change in the spectrum. H_β and

H_γ suggest at times single bright lines centrally placed on the broad absorption bands. The helium lines are weak and nebulous, and the line at λ 4471·7 is generally equal to or a trifle fainter than the magnesium line at λ 4481·3. "K" is very weak or absent in all the photographs.

Moore does not include this star in his Tables, and neither does Merrill give any record of a bright H_α.

In my previous paper on "Recent Observations of some Bright-hydrogen-line Stars," under the heading of this star on p. 416, the last paragraph should be deleted, as the observations there quoted refer to "Pleione" (28 Tau.) and not "Atlas" (27 Tau.).

H.D.C. 25940 (o Per.) $4^h\ 1^m\cdot4 + 47°\ 27'$ Mag. 4·03 Sp. B3neϵ.

Last photograph 1924 February 23. Since then 15 negatives have been secured, covering the dates 1924 November 3 and 1927 October 6.

In this spectrum H_β is always a very close double bright line on a broad absorption band, with a tendency for the red component to be brighter than the violet component. In H_γ and H_δ the bright components are always well seen with the red stronger than the violet component: the bright red component of H_ϵ when seen is always stronger than the violet component. The helium line at λ 4471·7 is always very strong and much more intense than the magnesium line at λ 4481·3, which is moderately strong also. "K" is always weak. The double bright ionised metallic lines seem to be absent.

The new series of photographs exhibit no change in the spectrum of any definite character.

Moore (146) inserts this star in his Table I.

Merrill (11) gives the intensity of bright H_α as 4 on a scale of 5 as maximum, from three negatives taken between 1920 September 18 and 1922 February 17.

H.D.C. 37202 (ζ Tau.) $5^h\ 31^m\cdot7 + 21°\ 5'$ Mag. 3·0 Sp. B3neγ.

Last photograph 1924 March 10. Since then 24 negatives have been secured between the dates 1924 September 2 and 1928 March 10 inclusive.

The characteristic appearance of the spectrum of this star has been described previously in my two earlier papers entitled "The Spectrum of ζ Tauri" (*M.N.R.A.S.*, **83**, 326) and "Recent Observations of some Bright-hydrogen-line Stars" (*M.N.R.A.S.*, **84**, 417), and these involved the discussion of all photographs up to 1924 March 10.

The new photographs have indicated distinct changes in the appearance of H_β, due either to the variable intensity of the central absorption line or to the variable intensities of the bright components which lie one on each side of it.

Further, the line "K" seems also to be variable in harmony with the changes of H_β.

A complete discussion of all the photographs of this star, numbering altogether 71, with reference to these and possibly other lines is in hand.

Plate 13, spectra numbers (2) and (3), illustrate the two extreme appearances of H_β, the former taken on 1923 February 14 showing the weak and narrow absorption line with the adjacent strong bright lines on each side of it, and the latter taken on 1927 October 7 exhibiting the strong and broad central absorption line with only faint impressions of the adjacent bright lines.

Moore (170) places this star in his Table III., giving Adams' period of 138 days.

Merrill (11) gives the intensity of H_α as 3 on a scale of 5 as maximum, from three negatives taken on 1920 February 7, 1921 February 7, and 1921 November 1.

H.D.C. 45542 (ν Gem.) $6^h\ 23^m{\cdot}0 + 20°\ 17'$ Mag. 4·06 Sp. B5neβ.

Last photograph 1924 March 16. Since then 9 more negatives have been taken between the dates 1924 November 27 and 1926 February 15.

The spectrum shows no apparent change. H_β, H_γ, and H_δ exhibit narrow absorption lines on broad absorption bands, and H_β sometimes seems to have two faint bright components of which the violet is slightly stronger than the red. The helium line at λ 4471·7 is moderately strong and more intense than the magnesium line at λ 4481·3. "K" is generally moderate to weak.

Moore (172) places this star in his Table III., referring to Harper's period of 9·6 years.

Merrill does not refer to the intensity of bright H_α, but in an earlier paper * he pointed out that, in 1915 November and December, "a feeble bright H_α is seen . . . apparently superposed on a broad weak absorption line. . . ."

H.D.C. 58715 (β CMi.) $7^h\ 21^m{\cdot}7 + 8°\ 29'$ Mag. 3·09 Sp. B8neγ.

This star was not included in the earlier series of observations of the bright-hydrogen-line stars. It was found, however, that in a photograph taken at Sidmouth on 1921 April 5, that H_β showed a distinct double bright line on a broad absorption band, and that H_γ also gave traces of these bright components. The star was therefore added to the working list for further observation.

Since the above negative was taken 11 others have been secured included in the dates 1923 January 10 to 1928 March 25. Most of these negatives exhibit H_β as showing well-separated double bright lines on a broad absorption band, and a few negatives the line H_γ likewise: H_δ and H_ϵ appear as narrow absorption lines on broad absorption bands.

The relative intensities of the bright components in each of the lines H_β and H_γ are about equal, but the negatives suggest that a variability may be present. All the helium lines in this spectrum are very weak,

* *Pub. Astr. Obs. Univ. Michigan*, **2**, 181, 1916.

and the magnesium line at λ 4481·3 is also weak but a little stronger than the helium line at λ 4471·7. The " K " line is absent.

The earliest definite observation of a bright hydrogen line other than H_α being seen is, so far as can be found, that recorded by Frost, Barrett, and Struve * in their investigation on the radial velocities of 368 helium stars. They secured negatives of the spectrum of this star between the dates 1904 November and 1911 November, and they state that " two of our plates show distinctly a bright line at H_β superposed over wide absorption."

Merrill † in describing the spectrum of this star stated :—

" Photographs taken on 1915 December 3 and 10 show a bright H_α line in the spectrum of this star, superposed on weak absorption. As the other hydrogen lines are dark it should also be included in the Electra group."

He further stated that " several hydrogen lines show traces of structure similar to that described for ν Geminorum," which star, according to him, displayed wide diffuse absorption lines with narrow central cores, the effect, as he suggested, of perhaps invisible bright components.

The Sidmouth photographs show clearly the bright components of the hydrogen lines H_β and H_γ, and it is quite possible that these bright lines have considerably increased their intensity since Merrill's observations were made, and have thus become easily observable.

Moore does not include this star in any of his tables.

Merrill (16) does not give a value for the intensity of bright H_α, but states that it was " bright " on 1921 March 29.

H.D.C. 109387 (κ Dra.) $12^h\ 29^m{\cdot}2 + 70°\ 20'$ Mag. 3·88 Sp. B5neγ.

Last photograph 1921 April 4. Since then 29 more negatives have been secured, covering the period 1924 June 25 to 1928 April 19.

In the spectrum of this star all the hydrogen lines are broad absorption bands. On each of the bands of the H_β and H_γ lines are double bright lines with the components more or less equal in intensity, and H_δ and H_ϵ show narrow absorption lines only in the middle of the broad bands. The helium absorption-lines are all fairly strong and nebulous, and the line at λ 4471·7 is always stronger than the magnesium line at λ 4481·3. The line " K " is always very weak or absent. No other bright lines than those of hydrogen can be seen with certainty. None of the new photographs show any marked change in the spectrum. Sometimes the red and violet components of the H_β and H_γ suggest a variation in relative intensity, but this cannot be said to be definite.

Robert H. Baker ‡ in 1920 found a short period of 8·986 days from measurements of the broad nebulous helium line at λ 4471·7 and the emission band at H_β, but he noticed that the narrow hydrogen absorption line appearing on this bright band did not share in this period. He

* *Ap. J.*, **64**, 24, 1926.
† *Pub. Astr. Obs. Univ. Michigan*, **2**, 181, 1916.
‡ *Ibid.*, **3**, 38.

suggests that they possibly have a periodic variation of about four years : such a variation is not apparent from the Sidmouth observations.

It is interesting to note that the narrow H_β absorption which was frequently measured by Baker on his early plates is stated by him to have diminished in width during the interval 1912 to 1915, and therefore was not seen by him. In all the Sidmouth photographs this central absorption line is clearly visible, so that it has most probably broadened out again.

Frost, Barrett, and Struve * found that their observations made between 1904 January and 1925 February did not satisfactorily represent Baker's orbit. This they say may in part be due to errors of measurement which are exceptionally large because of the presence of variable hydrogen emission lines.

Moore (174) places this star in his Table III., giving Baker's orbit.

Merrill (12) gives bright H_α the intensity of 4 on a scale of 5 as maximum, from three negatives taken on 1920 March 19, April 24, and July 12.

Plate 13, spectrum number 4, shows the spectrum of this star as photographed on 1927 June 3. H_β has clearly double bright components with a thin absorptive line dividing them, and H_γ fainter bright components with a stronger central absorption.

H.D.C. 200120 (f^1 Cyg.) $20^h\ 56^m{\cdot}4 + 47°\ 8'$ Mag. 4·86 Sp. Boneγ.

Last photograph 1923 November 11. Since then 4 other photographs have been secured between the dates 1924 August 18 and 1925 June 29, two being taken in each year. These photographs still show the continued reduction in intensity of the bright hydrogen lines, H_β and H_γ appearing as broad absorption bands with faint single bright lines down their centres.

Moore (161) places this star in his Table I.

Merrill makes no reference to the intensity of H_α.

H.D.C. 202904 (v Cyg.) $21^h\ 13^m{\cdot}8 + 34°\ 29'$ Mag. 4·42 Sp. B3neζ.

Last photograph 1923 November 30. Since then 11 other negatives have been secured between the dates 1924 June 30 and 1926 July 11. This spectrum does not seem to have undergone any alteration since the 1923 photographs. H_β, H_γ, and H_δ exhibit strong double bright lines on very broad absorption bands, and H_ϵ and H_ζ the same to a lesser degree of intensity. The violet components are always stronger than the red components. The nebulous helium absorption lines are strong, and that at λ 4471·7 is always very much stronger than the magnesium line at λ 4481·3, which is generally very weak.

Moore does not mention this star in any of his tables.

Merrill (13) gives the intensity of bright H_α as 3 on a scale of 5 as maximum, from a negative taken on 1920 September 12.

* *Ap. J.*, **64**, 25, 1926.

H.D.C. 209409 (o Aqr.) $21^h\ 58^m{\cdot}1 - 2°\ 38'$ Mag. 4·66 Sp. B5neγ.

Last photograph 1923 September 5. Since then 5 more negatives have been secured between the dates 1924 July 26 and September 1. None of the new photographs are very satisfactory, so that no definite conclusions can be drawn.

Moore does not mention this star.

Merrill (13) gives the intensity of bright H_α as 3 on a scale of 5 as maximum, from a negative taken in 1920 October 4.

H.D.C. 212076 (31 Peg.) $22^h\ 16^m{\cdot}6 + 11°\ 42'$ Mag. 4·93 Sp. B3neϵ.

Last photograph 1923 November 11. Since then 3 photographs have been secured between the dates 1924 August 7 and September 3. None of the later photographs are very satisfactory, the magnitude being rather low for this instrument. They show H_β as a strong single or very close double line on a broad absorption band, and H_γ, H_δ, and H_ϵ as bright or probably double bright lines on broad absorption bands. The helium lines are all strong and not so nebulous as in other stars of this class, and the magnesium line at λ 4481·3 is very weak. The "K" line is moderately strong and very sharp. No noticeable change can be detected in the spectrum.

Moore does not record this star in his tables.

Merrill (13) gives the intensity of bright H_α as 1 on a scale of 5 as maximum, from a negative taken on 1921 November 1. The strength of the bright hydrogen lines now recorded suggests that H_α must now be much stronger than intensity 1, and therefore that the spectrum is variable.

H.D.C. 212571 (π Aqr.) $22^h\ 20^m{\cdot}2 + 0°\ 52'$ Mag. 4·64 Sp. B1neδ.

Last photograph 1923 September 16. Since then 4 photographs have been taken between the dates 1924 July 26 and August 25.

H_β, H_γ, and H_δ show well-defined double bright lines on the broad absorption bands, the violet components being stronger than the red. The helium lines are nebulous and moderate in strength, and the magnesium line at λ 4481·3 is weak. The "K" line is moderate in intensity and appears sometimes to have bright borders.

The double bright ionised metallic lines which appear strongly in ϕ Persei are similar in appearance in this star, but are not quite so strong.

Moore does not mention this star in his tables.

Merrill (13) gives the intensity of bright H_α as 4 on a scale of 5 as maximum, from a negative taken on 1921 October 4.

H.D.C. 217891 (β Psc.) $22^h\ 58^m{\cdot}8 + 3°\ 17'$ Mag. 4·58 Sp. B5neϵ.

Last photograph 1923 September 11. Since then 5 other negatives have been taken between the dates 1924 July 26 and 1925 October 14.

In the spectrum of this star H_β and H_γ exhibit exceedingly strong

bright lines centrally placed on the broad absorption bands: they are most probably close doubles. H_δ and H_ϵ show the bright lines separated, the components being about of equal intensity. The helium lines are moderate in intensity, but not so nebulous as in many other stars of this class. "K" is very weak.

The double bright ionised metallic lines are here only faintly apparent.

The new negatives exhibit no definite changes from the earlier photographs.

Moore does not include this star in his tables.

Merrill (13) gives the intensity of bright H_α as 3 on a scale of 5 as maximum, from two negatives taken on 1919 June 30 and August 4.

Plate 13, spectrum number 5, is a reproduction of a photograph taken on 1924 July 26 and shows clearly the particulars of the hydrogen lines stated above.

Summary.

This paper is a continuation of the spectroscopic study of certain bright-hydrogen-line stars during the period 1924 March to 1928 April, following a previous study made from 1923 September to 1924 February.

All the new photographs have been made with the same instrument as formerly, and so are all comparable with one another.

The present observations have indicated distinct changes in the spectra of some of these stars, these being indicated under the respective star headings.

Attention may be called to the changes noted in the spectra of ϕ Persei, H.D.C. 20336 (– Camel.), 23 Tauri, ζ Tauri, and β Canis Minoris.

Description of the Plate.

The spectra of four bright-hydrogen-line stars compared with the helium star γ Orionis (No. 1).

Nos. 2 and 3 are spectra of ζ Tauri, showing the change in the appearance of H_β on the dates 1923 February 14 and 1927 October 7 respectively.

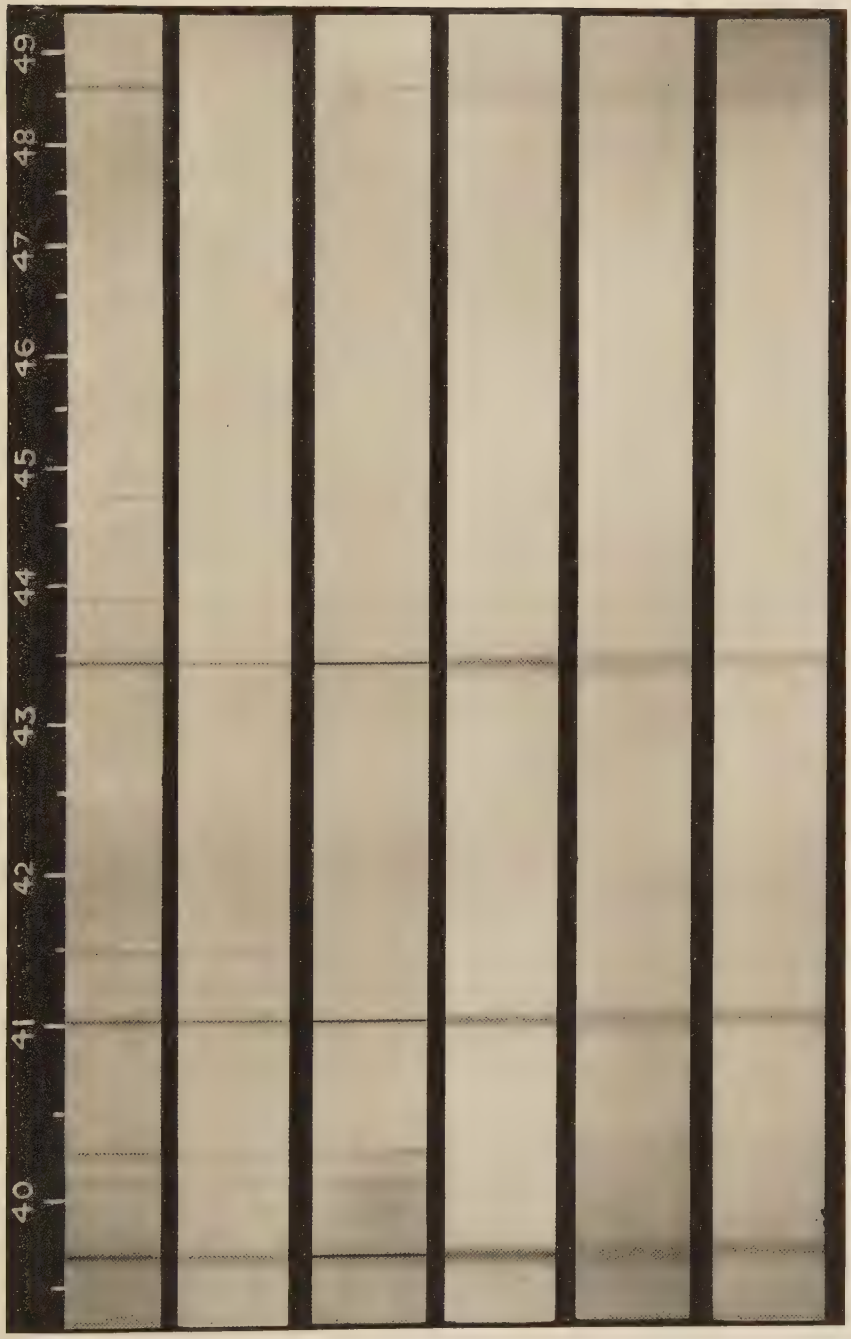
Nos. 4, 5, and 6 display the appearances of the hydrogen lines in the stars κ Draconis, β Piscium, and 23 Tauri respectively.

Further Observations of some Bright-hydrogen-line Stars, 1924–1927.

Addendum (May 28).

In my second paper on the spectrum of ϕ Persei (*M.N.R.A.S.*, **86**, 496) I stated that a conspicuous feature of all the negatives of this star was that "between the ionised iron double at $\lambda\lambda$ 4173·5 and 4178·9, and the ionised iron line at λ 4238·8, this region shows almost continuous homogeneous absorption."

This region of absorption seems apparently to be common to most of the bright-hydrogen-line stars, as can be seen by an examination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of the plate showing a series of spectra illustrated in my earlier paper on those stars (*M.N.R.A.S.*, **84**, Plate 9).

This feature is again strongly noticeable in the new series of spectra, as can be seen in the accompanying illustration (Plate 13); it is also conspicuous in the spectrum of γ Orionis, which is a pure absorption spectrum only.

No origin has at present been found for this interesting peculiarity.

Note on a Correction to Spectroscopic Parallaxes. By D. L. Edwards, A.R.C.S., D.I.C.

(*Communicated by the Director of the Norman Lockyer Observatory*)

The presence of spectroscopic binaries forms a source of error in the determination of parallaxes by the spectroscopic method which in the majority of cases has been either wholly or partially ignored, owing to the lack of knowledge concerning the relative brightness of the components in such binary systems. Dr. H. C. Woods * has applied a correction to the apparent magnitude of those spectroscopic binaries which are described in the Lick Observatory list † as showing two spectra; Professor Adams and his collaborators,‡ and Messrs. Young and Harper,§ have also applied corrections in an apparently arbitrary manner to the magnitudes of some of the spectroscopic binaries in their lists; many observers, however, have not applied any correction except in the cases where the individual magnitudes of the components are known. The nature of the methods employed in obtaining spectroscopic parallaxes necessitates an assumption of some kind, either explicitly or otherwise, with regard to the magnitudes of spectroscopic binaries. In cases where such binaries are treated as ordinary single stars, or where no attempt is made at correction, the assumption is implied that the difference in magnitude between the components in each system is considerable—*i.e.* that the secondary emits a negligible fraction of the combined light of the pair. The objects of the present note are to suggest, as an alternative, an apparently more reasonable assumption; to test this assumption by a comparison of published spectroscopic parallaxes with standard trigonometric and group parallaxes; and to deduce a series of corrections to existing spectroscopic parallaxes which will reduce the errors in the latter due to the presence of spectroscopic binaries.

The assumption suggested is that the difference in magnitude between the components of a spectroscopic binary, instead of being

* *M.N., R.A.S.*, **87**, 387.
† *L.O.B.*, 355.
‡ *Ap. J.*, **53**.
§ *Pub. D.A.O.*, **3**, No. 1.

very large, is equal to some suitably chosen constant value, such as the average value of this difference for visual binaries. Since the spectrum observed for the purpose of parallax work may be taken to be that of the primary (apart from exceptional cases in which the two components are of almost equal magnitudes), the apparent magnitude of the primary is required, and may be found from the known total magnitude of the system by the addition of a constant quantity whose value depends on the assumed difference of magnitude. This was the method adopted by Woods in the case of spectroscopic binaries showing two spectra. There seems, however, to be no justification for this limitation in applying the correction, since (as will be shown later) there still remains an error due to the uncorrected binary magnitudes. The question next arises as to the best value to adopt for the assumed magnitude difference. The consideration of a number of visual binaries taken from Aitken's list in his book, *The Binary Stars*, gives a mean value of $\Delta m = 0{\cdot}9$, and if this is taken as applicable also to spectroscopic binaries, the correction required to reduce the total magnitude to the magnitude of the primary is 0·4. In other words, if m is the known combined magnitude of a system, m_1 the magnitude of the primary, then

$$m_1 = m + 0{\cdot}4.$$

The corresponding correction used by Woods is 0·5 ; the mean value of the corrections used at Mount Wilson is 0·37 ; and Young and Harper's mean value is 0·54. The value 0·4 gives the best agreement with observed errors in the cases to be considered later, and has been adopted in this note.

The effect of ignoring this correction is to introduce two sources of error. The first of these arises in the formation of the standard reduction curves or tables connecting spectroscopic criteria with absolute magnitude. Among the standard stars of known parallax there will be a number of spectroscopic binaries, and for each of these the assumed apparent magnitude (and hence the absolute magnitude) will be too small by the amount 0·4. If there are n spectroscopic binaries among the total N standard stars, then (neglecting other sources of error) the assumed value of M will be correct for $N - n$ stars, and for n stars it will be 0·4 magnitude too bright. The adopted reduction curve will thus be displaced from its true position by an amount $0{\cdot}4\nu$ along the M-axis, where ν is the ratio n/N, provided that the frequency distribution of spectroscopic binaries is independent of absolute magnitude (*i.e.* that they are evenly distributed throughout the curve). In other words, if M is the absolute magnitude as found from the uncorrected curve, and M_0 is the true value, then

$$M_0 = M + 0{\cdot}4\nu.$$

This correction applies to all the stars, whether binaries or not, and leads to a correction to the parallax of the form

$$\pi_0 = \pi \times 1{\cdot}2^{\nu}.$$

The second source of error occurs in deducing parallaxes from the absolute magnitudes, owing to the erroneous values adopted for m. This will only affect spectroscopic binaries, since for other stars m is correct. For the former we have (incorporating both corrections)

$$\log \pi_0 = \frac{M + 0{\cdot}4\nu - (m + 0{\cdot}4) - 5}{5} = \log \pi + 0{\cdot}08(\nu - 1),$$

giving

$$\pi_0 = \pi \times 1{\cdot}2^{(\nu-1)}.$$

Some actual cases will next be considered, commencing with the simplest, in which trigonometric parallaxes alone have been used as standards, and no correction has been made to the magnitudes of spectroscopic binaries. If π is the spectroscopic parallax, π_0 the true value and ϖ the trigonometric, then it may be assumed that $\overline{\varpi} = \bar{\pi}_0$ if a sufficiently large number of stars are taken to neglect accidental errors. It is first of all necessary to render the spectroscopic parallaxes directly comparable with the trigonometric values by the elimination of any systematic difference. This may be done by the determination of a correction κ equal to the mean value of ϖ/π. The quantity $\kappa\pi$ will then differ from ϖ only on account of accidental errors and of the spectroscopic binary errors. (The latter will have no systematic effect on the series taken as a whole.) If now the spectroscopic binaries alone are considered, and there are a sufficient number of these to neglect accidental errors, then the mean value for these stars of the quantity $\varpi/\kappa\pi$ will give the correction required for which the theoretical value of $1{\cdot}2^{(\nu-1)}$ has been found above. Similarly, the mean value of $\varpi/\kappa\pi$ for non-binary stars will give the correction factor required for these stars, corresponding to the theoretical value of $1{\cdot}2^{\nu}$. Owing to uncertainty in radial velocity measurements or to lack of observations, a small percentage of the stars will be placed in the wrong category, so that the above relations will not be exact; the following results, however, seem to indicate that the approximation is sufficiently close.

Rimmer's Parallaxes of Late-type Stars.—In Rimmer's first paper * there are 272 standard stars of known trigonometric parallax, of which 60 are spectroscopic binaries. This gives theoretical values of the corrections to be $1{\cdot}2^{\nu-1} = 0{\cdot}87$ and $1{\cdot}2^{\nu} = 1{\cdot}04$. Now the actual value of $\Sigma\varpi/\pi = 283{\cdot}4$ for 272 stars, giving $\kappa = 1{\cdot}04$. For the 60 spectroscopic binaries alone $\Sigma\varpi/\pi = 54{\cdot}17$, so that the observed value of the correction to be applied to these stars is $54{\cdot}17/60{\cdot}\kappa = 0{\cdot}87$. Similarly the correction to non-binary stars is 1·03, and the agreement with the above theoretical values is almost exact.

It is possible at this stage to derive a partial check on the original assumptions by reversing the process, and assuming that the observed values of the corrections, viz.

$$\frac{\Sigma_a \varpi/\pi}{n\kappa} \quad \text{and} \quad \frac{\Sigma_b \varpi/\pi}{(N - n)\kappa}$$

* *Memoirs R.A.S.*, **62**, 113.

for the binaries and non-binaries respectively (called a and b in the following equations) are of the form

$$a = x^{p-1} \quad \text{and} \quad b = x^{p},$$

from which an empirical value of x ($= b/a$) may be determined. The agreement, or otherwise, with the theoretical value 1·2 will serve as a check on the assumed average magnitude difference upon which this value is based. The value of x for the stars in the above paper of Rimmer is 1·19, which seems to be a satisfactory agreement.

*Adams and Joy, A-type Stars.**—In this case the number of standard stars is 184, including 64 spectroscopic binaries, so that

$$\nu = 0{\cdot}348, \quad 1{\cdot}2^{\nu-1} = 0{\cdot}89, \quad \text{and} \quad 1{\cdot}2^{\nu} = 1{\cdot}07.$$

Among the standard stars both group motion and trigonometric parallaxes are used, and these may be considered either separately or together in deducing the corrections a and b. For the cluster stars, the factor κ required to correct for the systematic difference between spectroscopic and cluster parallaxes is 1·08, and the values of a and b are 0·90 and 1·07 respectively. For trigonometric parallaxes only, $\kappa = 0{\cdot}99$, $a = 0{\cdot}92$, and $b = 1{\cdot}04$. For all the stars taken together, $\kappa = 1{\cdot}03$, $a = 0{\cdot}91$, $b = 1{\cdot}05$, and the value of x derived from these results is 1·15.

The above examples are sufficient to show the methods adopted. The results, together with those from other spectroscopic parallaxes, are summarised in the table on p. 699. In some cases theoretical values of the corrections are indeterminate, or take a more complicated form, as, for example, when parallactic motions or other methods are used in forming the standard curves. In such cases the values of $1{\cdot}2^{\nu}$ and $1{\cdot}2^{\nu-1}$ are omitted. In those cases where the magnitudes of some spectroscopic binaries have been corrected by the authors, the individual stars so corrected have been treated as non-binaries for the purpose of deducing the theoretical corrections, and have been omitted entirely in computing the empirical values a and b. Cepheids, pseudo-cepheids, and giant M-type stars have also been excluded.

Consideration of this table shows, firstly, that in each of the eleven cases which have been examined the spectroscopic parallaxes of spectroscopic binaries are, in the mean, too large, while those of single stars are too small. These spectroscopic parallaxes may be brought into better agreement with the standard parallaxes by applying the correction factors a and b to spectroscopic binaries and single stars respectively. There is good agreement between these factors and their theoretical values $1{\cdot}2^{\nu-1}$ and $1{\cdot}2^{\nu}$ which seems to justify the assumptions upon which the latter are based.

If there are present any undetected spectroscopic binaries, the effect will be to make the quantities $1{\cdot}2^{\nu-1}$, $1{\cdot}2^{\nu}$, and b too small, while a will remain practically unaffected. It is interesting to note that while $1{\cdot}2^{\nu-1}$, as should be expected, is generally slightly smaller than a, yet

* *Ap. J.*, **56**, 242.

the mean difference is very small (0·02), which appears to indicate that there are very few undetected spectroscopic binaries among the stars in these lists. Since b is too small on account of these undetected binaries, the quantity x will also be too small, though only very slightly. The actual value of $\bar{x}$ is 1·18, and the assumed theoretical value of 1·2 (implying a mean magnitude difference of 0·9 between the components in binary systems) is thus strongly supported.

Source.	N.	κ.	$1{\cdot}2^{\nu-1}$.	$1{\cdot}2^{\nu}$.	a.	b.	x.	Refs.
Mt. Wilson, 1646 stars	648	1·05	0·85	1·02	0·87	1·01	1·16	*Ap. J.*, **53**, 13.
Rimmer, 1st list	272	1·04	0·87	1·04	0·87	1·03	1·19	*Mem. R.A.S.*, **62**, 113.
Rimmer, 2nd list	92	1·04	0·87	1·04	0·85	1·03	1·21	*Ibid.*, **64**, 1.
Young and Harper	353	0·97	0·85	1·02	0·87	1·02	1·17	*Pub. D.A.O.*, **3**, 1.
Mt. Wilson, A-types	..	..	..	..	..	..	..	*Ap. J.*, **56**, 242.
Group motion only	82	1·08	..	..	0·90	1·07		
Trig. only	102	0·99	..	..	0·92	1·04		
All stars	184	1·03	0·89	1·07	0·91	1·05	1·15	
Miss Douglas, A-types	..	..	..	..	..	..	..	*Ap. J.*, **64**, 262.
Group motion only	28	1·07	..	..	0·95	1·08		
Trig. only	46	0·99	..	..	0·86	1·12		
All stars	74	1·01	0·92	1·10	0·91	1·10	1·21	
Abetti, A-types	93	1·03	0·90	1·08	0·90	1·08	1·20	*Mem. Arcetri*, fasc. 42.
Woods (uncorrected stars)	131	0·96	0·88	1·06	0·96	1·02	1·06	*M.N.*, *R.A.S.*, **87**, 387.
Edwards, 1st method	58	1·20	..	..	0·97	1·04	1·07	*Ibid.*, **84**, 366; **85**, 439.
Edwards, 2nd method	66	1·12	..	..	0·93	1·07	1·14	*Ibid.*, **87**, 364.
Mt. Wilson, B-types	81	1·22	..	..	0·82	1·09	1·33	*Ap. J.*, **57**, 294.

A further refinement may be possible by separating spectroscopic binaries showing two spectra from those showing only one. The material is too scanty to admit of such separation in the individual lists of spectroscopic parallaxes, but the consistency of the values of x, indicating an approximate uniformity of Δm among these lists, seems to justify the grouping together of results from all the sources for this purpose. If this is done, the mean factors required to correct parallaxes of binaries showing one and two spectra respectively are

$$a_1 = 0{\cdot}92\,; \quad a_2 = 0{\cdot}84.$$

These are the mean values of $\varpi/\kappa\pi$, where κ takes its appropriate value according to the source. The theoretical values of these corrections take a slightly different form from that derived for all the spectroscopic binaries together. If r represents the quantity to be added to the apparent magnitude of a binary to reduce it to that of the primary (assumed above to be 0·4 in the mean), then

$$5 \log \pi_0 = \mathrm{M} + 0{\cdot}4\nu - m - r - 5 = 5 \log \pi + 0{\cdot}4\nu - r.$$

* Compared with group motion parallaxes only.

This is not strictly correct, since the quantity $0{\cdot}4\nu$ (due to the errors of the standard curves) in the case of Woods' parallaxes should be $r_1\nu$; in the case of Mount Wilson (late types) and Victoria it should lie between $r_1\nu$ and $0{\cdot}4\nu$; and in the remaining cases it is correct. The above equation, however, is a very close approximation, and the errors introduced by its use are negligible. We therefore have

$$\log a_1 = 0{\cdot}08\bar{\nu} - 0{\cdot}20r_1,$$

with a similar expression for $\log a_2$. The value of $\bar{\nu}$ is from all the stars in the different lists taken together, and

$$0{\cdot}08\bar{\nu} = \log \bar{b} = 0{\cdot}02.$$

So that r_1 and r_2 can be easily determined. The above values of a_1 and a_2 give $r_1 = 0{\cdot}3$, $r_2 = 0{\cdot}5$. This implies that the mean value of Δm (the difference in magnitude between the components) is 0·6 for spectroscopic binaries showing two spectra, and 1·3 for those showing one spectrum.

The mean values of the corrections applicable to spectroscopic parallaxes (apart from systematic corrections) are thus found to be 1·05 for non-binary stars, 0·92 for spectroscopic binaries showing one spectrum, and 0·84 for binaries showing two spectra. These corrections are in some cases unimportant or negligible, but in others they are quite appreciable, and the general accuracy of the results should be slightly improved by their use.

On the Low Temperature of the B1-type Star ζ Persei.

By H. Kienle.

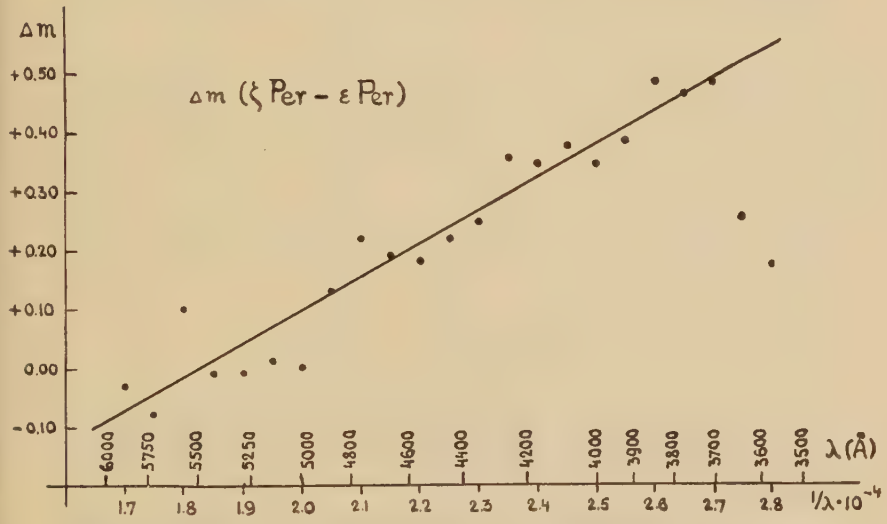
In the course of investigations on the continuous spectra of the stars I have been led to the discovery of some discrepancies between two stars of type B1, ζ and ϵ Persei. By comparing these stars, which appear on the same plate, it was found that the intensity-distribution of ζ Persei approximates to that of a star of spectral type A0. I have since seen a remark of Messrs. Greaves, Davidson, and Martin * that they also found ζ Persei to show an intensity-distribution which corresponds to low temperature. It seems therefore interesting to compare these results with ours.

In the case of the comparison of ζ and ϵ Persei it is difficult to see to what extent the results of Greaves, Davidson, and Martin are influenced by the systematic errors which may be introduced by their method. For the comparison must be made by way of ζ Per., β Ari., η Tau., η U. Ma., γ Cas., ϵ Per. But the fact that the colour indices of ζ and ϵ Per. show a remarkable difference is in favour of the results of Greaves, Davidson, and Martin. In our work ζ and ϵ Persei appear on the same

* *M.N., R.A.S.*, **87**, 352.

plate, with exactly the same exposure-time. The reductions are free from systematic errors ; for we determine the sensitivity-curve of each plate separately for the different wave-lengths by sets of standard stars with and without grating. A full description of the methods employed will be given in a forthcoming number of the publications of our Observatory. Here I only treat the comparison of ζ and ϵ Persei in order to test and confirm the statement of Greaves, Davidson, and Martin.

The diagram shows the intensity-differences of the two stars, plotted against wave-number. If we omit the two points at $1/\lambda = 2{\cdot}75.10^4 cm^{-1}$



and $2{\cdot}80.10^4 cm^{-1}$, the observations may be represented by a straight line with the gradient

$$\frac{d\Delta m}{d1/\lambda} = +0{\cdot}567 \pm 0{\cdot}030(p.e.).$$

The *p.e.* of one difference Δm measured is $\pm 0^m{\cdot}035$, which shows the high accuracy obtained. The scattering of the points for $1/\lambda < 2{\cdot}05$ is due to the fact that orthochromatic emulsions have a minimum of sensitivity in the green (λ 5000); this makes the measures in the region of the second maximum of sensitivity (near λ 5500) difficult to reduce.

We may now deduce the temperature of ζ Persei if that of ϵ Persei is known. Putting

$$\tau = \frac{1{\cdot}086}{1 - e^{-\frac{c_2}{\lambda T}}} \cdot \frac{c_2}{T},$$

a table has been constructed which contains the values of τ as functions of $1/\lambda$ and c_2/T or T. The value T_ϵ of ϵ Persei being given, we take

from the table for the mean value of $1/\lambda$ ($2{\cdot}20.10^4$) the value τ_ϵ, and have

$$\tau_\zeta = \frac{d\Delta m}{d1/\lambda} + \tau_\epsilon.$$

Then the same table gives us the value of T_ζ, which corresponds to the given values of $1/\lambda$ and τ_ζ. The following table contains the results on different assumptions as regards the black-body temperature of ϵ Persei :—

T_ϵ	τ_ϵ	τ_ζ	*p.e.*	T_ζ	*p.e.*
°				°	°
10,000	1·628	2·195	± 0·036	7,200	± 100
15,000	1·187	1·754		9,150	180
20,000	0·983	1·550		10,550	240
25,000	0·867	1·434		11,650	270
30,000	0·800	1·367		12,250	330

Greaves, Davidson, and Martin found the following values to correspond with each other :—

T_ϵ	T_ζ
°	°
13,800	7600
22,000	9100

The discrepancy in the black-body temperatures of the two stars is here somewhat greater than with us. But even with the most probable temperature for ϵ Persei (about 20,000°), that for ζ Persei only just reaches the temperature of normal A-stars.

The two points most to the right of the diagram may be explained as caused by the continuous hydrogen-absorption in the spectrum of ϵ Persei, which makes the intensities for this star too low as compared with black-body radiation. Since there is no sign of emission lines in the spectrum of ζ Persei, it is not probable that the intensity of ζ Persei should be greater than that of a black body of temperature T_ζ, as would be the case if there were continuous hydrogen-emission at the end of the Balmer series.

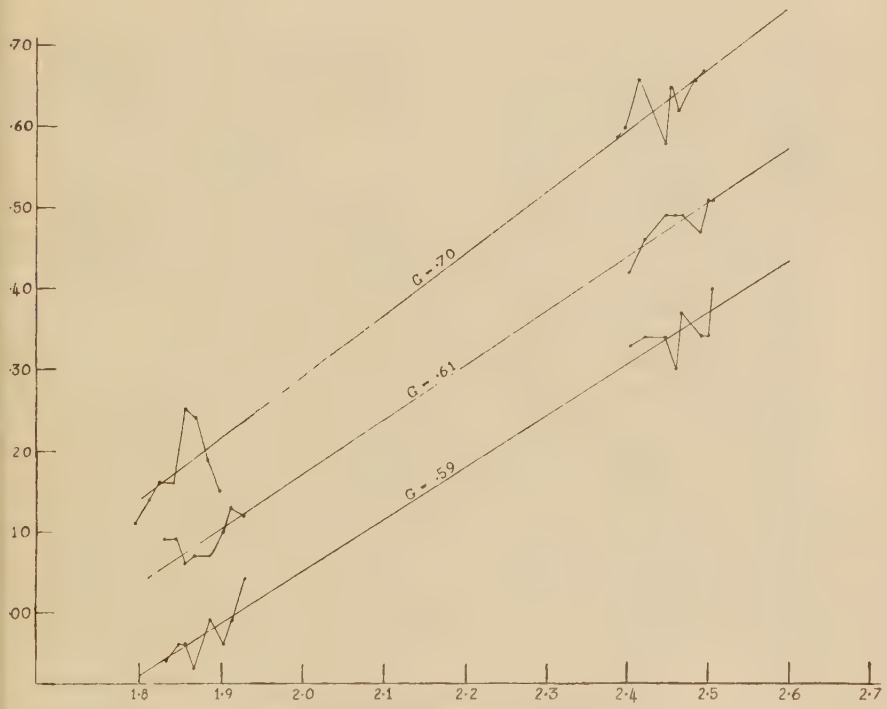
The line spectra of the two stars are in full agreement, and there should be no difference at all in spectral type as derived from the intensities of the absorption of H and He. The difference in the continuous spectra leads to the conclusion that the density of the atmosphere of ζ Persei must be remarkably lower than that of ϵ Persei.

Göttingen, Universitäts-Sternwarte :
1928 *May* 22.

The Low Colour-temperature of ζ Persei. (*Note to Dr. Kienle's Paper.*) By W. M. H. Greaves, C. Davidson, and E. Martin.

(*Communicated by the Astronomer Royal.*)

Since the publication of the results referred to by Dr. Kienle, further observations have been secured at Greenwich. Ten comparisons in-



Reciprocal of wave-length (in microns).

Gradients for ε Persei – ζ Persei.

volving ε Persei and ten involving ζ Persei are now available. These include three direct comparisons between these two stars, the "gradients" (*i.e.* the values of $0{\cdot}922 \times d\Delta m/d(1/\lambda)$) given by these three comparisons being + 0·60, + 0·57, and + 0·70, with a mean of + 0·62 in the sense ε Persei minus ζ Persei (the positive sign denoting a higher colour-temperature for the first star *).

A preliminary reduction gives the following values for the gradients of these stars when compared with an average star of type Ao. The

* The notation differs in sign from that of Dr. Kienle.

colour-temperatures T are derived on the hypothesis that the colour-temperature of an average A0 star is 13,000° K.

	Spectral Type.	Gradient.	T.
ε Persei	B1	+0·26	18,800
ζ Persei	B1	−0·40	9,100
	mean A0	(0·00)	(13,000)
	mean A5	−0·40	9,100

This gives + 0·66 for the gradient of ε Persei minus ζ Persei. The three direct comparisons give, as has been stated above, a mean gradient of + 0·62. The measurements for these three results are plotted in the figure, the actual quantities plotted being Δm (magnitude differences) against $1/\lambda$ (λ being measured in microns). The diagrams are in fact similar to that reproduced in Dr. Kienle's paper. It will be noticed that the measurements are confined to the extremities of the spectral range measured, experience having shown that the plot of Δm against $1/\lambda$ is a straight line, as is to be expected if stellar radiation is distributed in wave-length according to the "black-body" law. As it is our sole object to measure the slope of this line, it is more economical to confine our measurements to the extremities.

The measured gradients are + 0·59, + 0·61, and + 0·70. The corrections for selective atmospheric absorption were computed on the assumption that the correction to the observed gradient for the atmosphere at the zenith is 0·73,* and are + 0·01, − 0·04, and 0·00 respectively, thus giving the final values quoted above.

These results indicate that plates taken on different nights give gradients which differ more than would be expected from the internal discordances for different wave-lengths on one plate. This point is shown clearly in the diagram.

The abnormally low colour-temperature of ζ Persei is now quite definitely established.

Observations of Novæ, 1927–28. By B. M. Peek.

Photometric observations of the following four Novæ were continued at Herne during the season 1927–28.

1. *Nova Persei* (1901).—The number of observations is smaller than usual owing to unfavourable skies, which greatly interfered with the continuity of the series, particularly toward the close of 1927. Magni-

* See *M.N., R.A.S.*, **87**, No. 5, p. 357, 1927 March.

tudes of comparison stars as in former years. Instruments, chiefly the 12¼-in. reflector, occasionally the 150-mm. refractor.

TABLE I.

Nova Persei (1901).

Observed on 74 Nights.

J.D. [2425]	Mag.	J.D. [2425]	Mag.	J.D. [2425]	Mag.
088·55	13·5±	150·38	13·28	233·26	13·52
095·54	13·47	151·47	13·50	257·24	13·28
096·49	13·53	156·43	13·77	261·26	12·98
097·55	13·77	157·41	13·54	263·43	12·91
100·46	13·62	158·52	13·84	273·36	13·04
102·56	13·74	159·52	13·77	276·36	13·04
103·44	13·74	162·42	13·72	278·31	13·11
104·46	13·67±	163·45	13·82±	280·34	13·08±
109·46	13·47	169·30	13·77	282·41	13·13
111·44	13·37	171·44	13·37	283·34	13·18
112·44	13·60	172·36	13·80	290·35	13·05
114·44	13·30	174·38	13·80	296·38	13·47
116·42	13·30	176·37	13·60	298·26	13·35
118·40	13·32	178·37	13·72	302·28	13·32
120·56	13·11	182·29	13·40	303·48	13·37
121·50	13·13	185·30	13·39	304·40	13·52±
125·44	13·11	189·30	13·82	316·35	13·44
126·44	13·13	197·41	13·25	322·31	13·42
131·46	13·42	198·29	13·40	326·34	13·32
135·42	13·37	204·44	13·35	329·37	13·60
142·42	13·35	206·50	13·67	330·43	13·60±
146·42	13·32	210·29	13·32	345·35	13·32
147·38	13·37	213·39	13·28	346·34	13·38
148·42	13·30	220·40	13·04	349·35	13·40
149·40	13·35	221·47	13·0±		

2. *Nova Ophiuchi* (1848).—It is rather remarkable that whereas in previous years maxima of about 12·1 magnitude have been recorded, in 1927 the star was never seen brighter than 12·4. This seems probably to be due to the discontinuity of the observations, and it is more than possible that maxima occurred during the gaps in the record. Comparison stars as formerly. Instrument, chiefly the 12¼-in. reflector.

TABLE II.

Nova Ophiuchi (1848).

Observed on 20 Nights.

J.D. [2425]	Mag.	J.D. [2425]	Mag.	J.D. [2425]	Mag.
003	12·45	088	12·75	118	12·55
023	12·5	095	<12·3	126	12·5
029	12·55	096	12·6	146	12·5
035	12·5	097	12·65	148	12·5
051	12·4±	112	12·6	150	12·45
066	12·45	114	12·55	157	12·5±
081	12·5	116	12·55		

3. *Nova Cygni* (1876).—No striking maxima were observed, but the star continued to show a small range of variability, some of which is probably real. Comparison stars as formerly. Instrument, the 12¼-in. reflector.

TABLE III.

Nova Cygni (1876).

Observed on 25 Nights.

J.D. [2425]	Mag.	J.D. [2425]	Mag.	J.D. [2425]	Mag.
023	<14·44	139	<14·44	176	14·71
029	<14·44	142	14·86	178	14·61
096	14·54	146	14·94	185	14·54
097	14·54	148	14·91	204	14·78
112	14·88	150	14·84	210	14·71
114	14·91	156	14·66	213	14·71±
116	14·91	157	14·66	257	14·71±
118	14·91	171	14·81		
126	14·81±	172	14·71		

4. *Nova Aquilæ* (1925).—This faint Nova, discovered by Max Wolf in 1925, continued to fade slowly. No comparison magnitudes appear to be available, but the star is kept under observation from time to time, and when last observed on October 24 was somewhat below 10th magnitude. A direct vision spectroscope used in conjunction with the 12¼-in. reflector shows that the spectrum is still an emission one, practically all the light being apparently concentrated in one line, so that the image in the spectroscope appears stellar, while the neighbouring stars are drawn out into lines.

Table IV., giving the equivalents of Julian and Civil dates, is inserted for convenience.

TABLE IV.

Civil Date.	Julian Date.	Civil Date.	Julian Date.
1927 May 1	[2425] 002	1927 Dec. 1	[2425] 216
June 1	033	1928 Jan. 1	247
July 1	063	Feb. 1	278
Aug. 1	094	Mar. 1	307
Sept. 1	125	Apr. 1	338
Oct. 1	155	May 1	368
Nov. 1	186		

The Definitive Elements of the Orbit of Comet 1925c (*Orkisz*).

By Professor N. Goryatschew.

(*Communicated by the Secretaries.*)

From 450 observations of this comet during the interval 1925 April–1926 May, allowing for perturbations from Mercury, Venus, Earth, Mars, Jupiter, and Saturn, I have found the following elements :

T = 1925 April 1·510610 U.T. ; e = 1·0006408 ; log q = 0·0451644 ;
θ = 318° 03′ 41″·7 ; i = 100° 00′ 57″·1 ; ω = 36° 11′ 40″·3
(Equinox 1925·0).

All observations were united as accurately as possible in 11 normal geocentric places :

			α 1925·0.	δ 1925·0.
			° ′ ″	° ′ ″
I.	1925	IV. 16·5 U.T.	339 49 13·54	+29 23 35·66
II.		V. 10·0	352 05 10·80	60 44 04·81
III.		V. 24·5	20 50 59·93	78 16 08·74
IV.		VI. 6·0	106 06 14·26	80 40 16·87
V.		VI. 26·5	146 55 42·30	66 29 52·84
VI.		VII. 20·0	158 13 01·55	54 18 04·26
VII.		VIII. 13·5	165 16 09·16	45 54 29·23
VIII.		XI. 20·0	179 42 30·91	35 16 01·39
IX.	1926	I. 17·5	169 41 05·59	41 17 14·74
X.		III. 6·5	149 07 18·10	42 57 34·00
XI.		IV. 18·5	139 06 19·61	38 29 07·26

The computation of these normal places with the above elements and perturbations gives the following residuals O–C in $\Delta\alpha \cos\delta$ and $\Delta\delta$ respectively :—

	$\Delta\alpha \cos\delta$.	$\Delta\delta$.		$\Delta\alpha \cos\delta$.	$\Delta\delta$.
	″	″		″	″
I.	−0·4	−0·3	VII.	−1·2	−1·5
II.	−1·0	+1·1	VIII.	+1·5	−0·1
III.	0·0	−1·1	IX.	+0·3	+0·2
IV.	−0·9	−1·9	X.	+0·7	−0·5
V.	0·0	+0·4	XI.	−1·1	+1·0
VI.	+3·0	+0·6			

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